SOME RESULT ON EXCIRCLE OF QUADRILATERAL

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Abstract

Any quadrilateral not necessarily have excircle, in this paper will discuss necessary and sufficient condition that any quadrilateral having excircle. It also will set the various lengths of the sides are formed from the construction result of excircle. Besides that we can also establish some other excircle and also be specified the length of radii and the relationship of the radii with the presence of excircle.

1. Background

In a triangle, can always be constructed incircle and excircle [3, 13], while in any quadrilateral, may not necessarily be formed incenter and the excenter. Several authors discuss the incircle of a quadrilateral is [4-7, 9, 11], the author only discusses about the comparative of radii, side and diagonal of the quadrilateral.

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At [1], the author discusses the excircle as in Figure 1.1, which if we look, then it is not an excircle, because each circle just touches one side and extension of the other two sides. Furthermore, [8, 9] to construct an excircle (as Figure 1.2), which gives the requirements that the quadrilateral $ABCD$ with sides $AB = a$, $BC = b$, $CD = c$ and $DA = d$, then $ABCD$ is tangential quadrilateral if $a + c = b + d$.

However [8] also explained that if $ABCD$ convex quadrilateral where opposite sides $AB$ and $CD$ intersect at $J$, and the sides $AD$ and $BC$ intersect at $K$ (see, Figure 1.3), then $ABCD$ is a tangential quadrilateral if and only if either of

$$BJ + BK = DJ + DK,$$

$$AJ - AK = CJ - CK.$$  

In addition to the [9] also indicated that if the $ABCD$ convex quadrilateral with sides $a$, $b$, $c$, $d$ has an excircle if and only if $|a - c| = |b - d|$ as well as to construct
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various subtriangles so it obtained four pieces circumradii order to obtain the various relationships between the fourth circumradii. If we look at Figure 1.2 above, of course, the most important first is to prove that the bisector \( \angle A \), external bisector \( \angle B \) and \( \angle D \) and opposite bisector \( \angle C \) are concurrence which then needs to be discussed how the radii of the excircle as well as a variety of side length resulting from the excircle constructing.

2. Theoretical Basis

In addition to calculating the radii of the circle tangent, can also be calculated distance of the center point to the one point of the outer angles of the triangle (external bisector). Josefsson in [7] lowers the formula in Lemma 2.1.

**Lemma 2.1.** If \( O \) is the center of the incircle, then \( \Delta ABC \) will apply

\[
\frac{OB^2}{ac} = \frac{s - c}{s - a},
\]

with is a semiperimeter.

**Proof.** See [10].

The constructed excircle on a convex quadrilateral produce concurrency of six bisector angle. To prove concurrency of six bisector angle, can be done in the following manner. Note Figure 2.1.

**Figure 2.1**

Note \( \Delta ABK \) in Figure 2.1. Since of the circle centered at \( E \) offending side of \( BK \) in point \( I \), the extension of \( AK \) at the point \( H \) and the extension of \( AB \) at point \( F \), the
circle is excircle of $\triangle ABK$. By making a line bisector angle of each $\angle A$, $\angle KBF$, and $\angle BKH$, then the third angle of bisector concurrence at the point $E$. Thus, all three lines of bisector are $AE$, $BE$, and $EK$ concurrence at point $E$.

Note $\triangle ADJ$. Since the circle centered at $E$ offend the extension side of $AD$ at point $H$, $AJ$ extension at point $F$ and $DJ$ at point $G$, the circle is excircle from $\triangle ADJ$. By creating a line bisector angle of each $\angle A$, $\angle JDK$, and $\angle DJF$ third-line of the bisector angle, namely $AE$, $DE$, and $EJ$ concurrence at point $E$.

Points $I$ and $G$ is a point of tangency of circle. By connecting the points $C$ and $E$ are formed two triangles, namely $\triangle CIE$ and $\triangle CEG$, so $\angle ICE = \angle GCE$ meaning that $CE$ is bisector line of $\angle KCJ$. Thus proven $AE$, $BE$, $CE$, $DE$, $EJ$ and $EK$ concurrence.

**Theorem 2.2.** An outer circle tangent quadrilateral with sides $a$, $b$, $c$ and $d$ has the length of radii

$$
\rho = \frac{L\square ABCD}{a - c} = \frac{L\square ABCD}{d - b}.
$$

**Proof.** See [10].

**Theorem 2.3.** Suppose the $\square ABCD$ with $AB = a$, $BC = b$, $CD = c$ and $AD = d$ and also has a tangential excircle. Then

$$
L\square ABCD^2 = abcd(\sin \gamma)^2,
$$

where $2\gamma$ is the number of opposite angles.

**Proof.** Consider Figure 2.2. Pull the line $BD$, so that there are two triangles that $\triangle ABD$ and $\triangle BDC$ at $\triangle ABD$ applies: $BD^2 = a^2 + d^2 - 2ad \cos \angle A$. And on the $\triangle BDC$ the same thing is applicable that $BD^2 = b^2 + c^2 - 2bc \cos \angle C$. 

Thus obtained
\[ a^2 + d^2 - 2ad \cos \angle A = b^2 + c^2 - 2bc \cos \angle C, \]
\[ (a^2 + d^2 - b^2 - c^2)^2 = 4(ad \cos \angle A - bc \cos \angle C)^2. \] (2.1)

Additionally, \( L_{ABCD} \) can be written as
\[ L_{ABCD} = L_{ABD} + L_{BCD} = \frac{1}{2} ad \sin \angle A + \frac{1}{2} bc \sin \angle C, \]
\[ 16L_{ABCD}^2 = 4(ad \sin \angle A + bc \sin \angle C)^2 \] (2.2)

add the equation (3.15) with (2.2) is obtained
\[ (a^2 + d^2 - b^2 - c^2)^2 + 16L_{ABCD}^2 \]
\[ = 4(ad \cos \angle A - bc \cos \angle C)^2 + 4(ad \sin \angle A + bc \sin \angle C)^2 \]
\[ = 4(a^2d^2 + b^2c^2) - 16abcd(\cos \gamma)^2 + 8abcd \]
\[ = (2ad + 2bc)^2 - (a^2 + d^2 - b^2 - c^2)^2 - 16abcd(\cos \gamma)^2 \]
\[ = [(2ad + 2bc) - (a^2 + d^2 - b^2 - c^2)][(2ad + 2bc) + (a^2 + d^2 - b^2 - c^2)] \]
\[ - 16abcd(\cos \gamma)^2 \]
\[ = 16(s-a)(s-b)(s-c)(s-d) - 16abcd(\cos \gamma)^2 \]
since \(a + b = c + d = s\) so

\[
16L\square ABCD^2 = 16abcd - 16abcd(\cos \gamma)^2,
\]

\[
L\square ABCD^2 = abcd(\sin \gamma)^2.
\]

3. Sides and Radii

3.1. Sides length

The constructed of excircle on the quadrilateral produce various sides. For more details, please see Figure 3.1.

![Figure 3.1](image)

In Figure 3.1, there exists \(\square ABCD\) with \(\angle A = 2\alpha\). Suppose the length of \(HK = e\) and \(FJ = f\). Since \(IK\) and \(HK\) are the tangents of the point \(K\), \(IK = HK = e\). Furthermore, \(FJ\) and \(GJ\) are also tangents from the point \(F\) so that \(GJ = FJ = f\). Let \(AH = x\). Then \(DK = x - d - e\).

Furthermore, since of \(AH\) and \(AF\) are tangents from point \(A\), \(AH = AF\), so \(BJ = x - a - f\). Since the \(DG\) and \(DH\) are also tangents from point \(D\), \(DG = DH\), so that \(DG = x - d\), which produces \(CG = x - c - d\).

Furthermore, since \(CI = CG = x - c - d\), the circle centered at point \(E\) is excircle from four triangles, namely \(\triangle ABK\), \(\triangle ADJ\), \(\triangle BCJ\) and \(\triangle CDK\). Thus the semiperimeter (\(s\)) of each triangle is
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\[ s_1 = s\Delta ABK \]
\[ = \frac{a + b + x - c - d + e + x - d - e + d}{2}, \]
\[ s_1 = \frac{a + b - c - d + 2x}{2}. \]

Since \( AH = AF = s\Delta ABK = \Delta ADJ = x \), call \( s_2 = s\Delta ADJ = x \), and then we have

\[ s_3 = s\Delta BCK \]
\[ = x - a, \quad (3.1) \]
\[ s_4 = s\Delta CDK \]
\[ = x - d. \quad (3.2) \]

By using Lemma 2.1 in \( \Delta ABK \) and equation (3.1),

\[ \frac{BE^2}{AB \times BK} = \frac{x - AB}{x - BK}, \]
\[ \frac{BE^2}{a(b + x - c - d + e)} = \frac{x - a}{x - (b + x - c - d + e)}, \]
\[ BE^2 = \frac{x - a}{c + d - b - e} a(b + x - c - d + e) \quad (3.3) \]

and in a similar way would be obtained

\[ EK^2 = \frac{e}{c + d - b - e} (x - e)(b + x - c - d + e), \quad (3.4) \]
\[ EJ^2 = \frac{f}{d - f} (x - f)(x - d + f), \quad (3.5) \]
\[ DE^2 = \frac{(x - d)}{d - f} d(x - d + f), \quad (3.6) \]
\[ JE^2 = \frac{f}{c + d - f - a} (x - a - f)(x - c - d + f), \quad (3.7) \]
\[ CE^2 = \frac{(x - a - b)}{(c + d - f - a)} b(x - c - d + f), \]  
(3.8)

\[ EK^2 = \frac{e}{c - e} (x - d - e)(x - c - d + e). \]  
(3.9)

By substituting the equation (3.4) to (3.8) is obtained

\[ e(x - e)(b + x - c - d + e)(c - e) = e(x - d - e)(x - c - d + e)(c + d - b - e), \]

\[ e = \frac{d(x - c)^2 - b(x - d)^2 - 2xd^2 - bcd + 2cd^2 + d^3}{(d^2 + cd - dx - bx)}. \]  
(3.10)

Furthermore to find the value of \( f \), substitution of equation (3.5) to the equation (3.7) is obtained

\[ f(x - f)(x - d + f)(c + d - f - a) = f(x - a - f)(x - c - d + f)(d - f), \]

\[ f = \frac{-cx^2 + ax^2 - 2adx + acd + ad^2}{ad + ac - cx - ax}. \]  
(3.11)

Note the \( \triangle AEH \). Since \( \angle HAE = \alpha \), it is obtained

\[ \frac{AH}{\sin(90^\circ - \alpha)} = \frac{EH}{\sin \alpha}, \]

\[ AH = \frac{EH}{\sin \alpha} \cos \alpha. \]

In a similar way also be obtained

\[ e = \frac{cd[-2L\square ABCD \cot \alpha + (a - c)(c - b - 2d)]}{L\square ABCD \cot \alpha(-b - d) + ad - bcd}. \]  
(3.12)

By substituting the equation (3.8) to the equation (3.7) is obtained

\[ f = \frac{acd(a - c - d)}{(L\square ABCD \cot \alpha(-a - c) + (a - c)(ad + ac))}. \]  
(3.13)
from $x$, $e$, and $f$ and $DH = x - d$, then

$$DH = \frac{L_{ABCD}}{a-c} \cot \alpha - d,$$

$$CI = \frac{L_{ABCD}}{a-c} \cot \alpha - a - b.$$

Furthermore, by using the principle of Pythagoras obtained

$$AE^2 = \left(\frac{L_{ABCD}}{a-c} \cot \alpha\right)^2 + \left(\frac{L_{ABCD}}{a-c}\right)^2$$

$$= \left(\frac{L_{ABCD}}{a-c}\right)^2 ((\cot \alpha)^2 + 1),$$

$$BE^2 = \left(\frac{L_{ABCD}}{a-c} \cot \alpha - a\right)^2 + \left(\frac{L_{ABCD}}{a-c}\right)^2$$

$$= \frac{[L_{ABCD} \cot \alpha - a(a - c)]^2 + L_{ABCD}^2}{(a - c)^2},$$

$$CE^2 = \left(\frac{L_{ABCD}}{a-c} \cot \alpha - a - b\right)^2 + \left(\frac{L_{ABCD}}{a-c}\right)^2,$$

$$DE^2 = \left(\frac{L_{ABCD}}{a-c} \cot \alpha - d\right)^2 + \left(\frac{L_{ABCD}}{a-c}\right)^2.$$

3.2. Radii of the other excircle

Note Figure 3.2. If incircle formed on $\Delta BCJ$ centered in $O_b$ and $\Delta CDK$ based in $O_c$, the circle is also an excircle. Thus the length of radii of excircle on $\square ABCD$ that offensive side of $b$ and $c$ can be solved by using excircle theorem for the triangle.
Figure 3.2

Note $\triangle BCJ$. Suppose $\angle BCD = 2\theta$. Then $\angle BCJ = 180^\circ - 2\theta$. So the length of the radii is symbolized by $R_b$ is

$$R_b = (s\triangle BCJ - BJ)\tan\frac{1}{2}(180^\circ - 2\theta)$$

$$= (BJ + JF - BJ)\tan(90^\circ - \theta)$$

$$= f \cot \theta,$$

$$R_b = \frac{acd(a - c - d)}{L\square ABCD \cot \alpha - a - c + (a - c)(ad + ac)} \cot \theta.$$ 

In a similar way would be obtained

$$R_c = \frac{cd[2L\square ABCD \cot \alpha + (a - c)(c - b - 2d)]}{L\square ABCD \cot \beta - b - d + ad - bcd} \cot \theta.$$ 

3.3. Relationship of radii of the other excircle

The following are given characteristics of quadrilateral that has circle tangent a second shape that associated with the circle tangent first form.

**Theorem 3.4.** Given a $\square ABCD$ has excircle if and only if $R_aR_b = R_cR_d$.

**Proof.** ($\Rightarrow$) Suppose a $\square ABCD$ has excircle with long of radii $R_a$, $R_b$, $R_c$ and $R_d$, and also has a excircle in front of the point of $C$. Will be shown that $R_aR_b = R_cR_d$. 

$$R_aR_b = R_cR_d.$$
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Note Figure 3.3: a $\square ABCD$ the circle is a circle that is centered on $O_a$ offensive in side $a$ at the point of $L$, $O_b$ offensive side $b$ at the point $M$, $O_c$ offensive side $c$ in point $N$ and $O_d$ that offensive side of $d$ at the point $P$. Suppose the length of radii of the circle is $R_a$, $R_b$, $R_c$ and $R_d$ each of which is the radii of the circle centered at $O_a$, $O_b$, $O_c$ and $O_d$.

The second form of excircle is the circles are in front of the point $C$. Let the circle of offending extension of sides $AB$, $BC$, $CD$ and $AD$, respectively at point $F$, $I$, $G$ and $H$. Thus, the radii of the circle are $\rho = OH = OI = OG = OF$. Note $\triangle AOF$, using trigonometric rules we obtained

$$AF = \rho \cot \frac{A}{2},$$

(3.14)

$$\angle OBF = \frac{180^\circ - B}{2}$$

$$= 90^\circ - \frac{B}{2}.$$  

So that,

$$\tan \angle OBF = \frac{\rho}{BF},$$

$$BF = \rho \tan \frac{B}{2}.$$  

(3.15)
By subtracting the equations (3.14) and (3.15) was obtained

\[ a = \rho \left( \cot \frac{A}{2} - \tan \frac{B}{2} \right). \]

In a similar way would be obtained

\[ AL = R_a \tan \frac{A}{2}, \quad (3.16) \]
\[ BL = R_a \tan \frac{B}{2}. \quad (3.17) \]

By adding equations (3.16) and (3.17) was obtained

\[ AL + BL = R_a \left( \tan \frac{A}{2} + \tan \frac{B}{2} \right), \]
\[ a = R_a \left( \tan \frac{A}{2} + \tan \frac{B}{2} \right). \quad (3.18) \]

By doing the same way to \( \Delta BOI \) and \( \Delta COI \) and \( \Delta BO_bM \) and \( \Delta CO_bM \) obtained

\[ b = R_b \left( \tan \frac{B}{2} + \tan \frac{C}{2} \right). \quad (3.19) \]

By doing the same way to \( \Delta DE_cG \) and \( \Delta CE_cG \) as well as \( \Delta CO_cN \) and \( \Delta DO_cN \) derived

\[ c = R_c \left( \tan \frac{C}{2} + \tan \frac{D}{2} \right). \quad (3.20) \]

By doing the same way to \( \Delta AOH \) and \( \Delta DOH \) and \( \Delta DO_dP \) and \( \Delta AO_dP \) obtained

\[ d = R_d \left( \tan \frac{A}{2} + \tan \frac{D}{2} \right) \quad (3.21) \]

and
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\[ pp \left( \cot \frac{A}{2} - \tan \frac{B}{2} \right) \left( \tan \frac{B}{2} - \cot \frac{C}{2} \right) \]

\[ = R_u R_b \left( \tan \frac{A}{2} + \tan \frac{B}{2} \right) \left( \tan \frac{B}{2} + \tan \frac{C}{2} \right) \]

\[ = R_u R_b \left( \frac{\sin \frac{A}{2} \cos \frac{B}{2} + \cos \frac{A}{2} \sin \frac{B}{2}}{\cos \frac{A}{2} \cos \frac{B}{2}} \right) \left( \frac{\sin \frac{B}{2} \cos \frac{C}{2} + \cos \frac{B}{2} \sin \frac{C}{2}}{\cos \frac{B}{2} \cos \frac{C}{2}} \right) \]

which will be equal to

\[ \frac{\cos \frac{A + B}{2} \left( -\cos \frac{B + C}{2} \right)}{\sin \frac{A}{2} \cos \frac{B}{2} \sin \frac{C}{2}} = R_u R_b \left( \frac{\sin \frac{A + B}{2} \sin \frac{B + C}{2}}{\cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}} \right). \]

\[ \frac{\rho \rho}{R_u R_b} = \frac{\sin \frac{A + B}{2} \sin \frac{B + C}{2}}{\cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}} \left( \frac{A + B}{2} \left( -\cos \frac{B + C}{2} \right) \right). \]

\[ \frac{\rho \rho}{R_u R_b} = -\tan \frac{A + B}{2} \tan \frac{B + C}{2} \tan \frac{A}{2} \tan \frac{C}{2}. \] \hspace{1cm} (3.22)

Return the same way will be obtained

\[ \frac{\rho \rho}{R_u R_d} = -\tan \frac{A + D}{2} \tan \frac{D + C}{2} \tan \frac{A}{2} \tan \frac{C}{2}. \] \hspace{1cm} (3.23)

since

\[ \tan \frac{A + B}{2} = -\tan \frac{D + C}{2}, \]

and

\[ \tan \frac{B + C}{2} = -\tan \frac{A + D}{2}. \]

So that

\[ \tan \frac{A + B}{2} \tan \frac{B + C}{2} = \tan \frac{A + D}{2} \tan \frac{C + D}{2}. \]
by equations (3.22) and (3.23) was obtained

\[
\frac{\rho_a \rho_b}{R_a R_b} = \frac{\rho_c \rho_d}{R_c R_d}
\]

so

\[
R_a R_b = R_c R_d.
\]

(⇐) Suppose \( R_a R_b = R_c R_d \) will be shown that \( \square ABCD \) has excircle. Note Figure 3.4. Suppose the point of tangency of the circle centered at the point \( O_a, O_b, O_c \) and \( O_d \) that points \( L, M, N \) and \( P \).

![Figure 3.4](image)

**Figure 3.4**

\( E_a \) point is the center point formed by the intersection of each bisector \( \angle A \) and \( \angle CBJ \). \( E_b \) point is the center point formed by the intersection of each bisector \( \angle C \) and \( \angle CBJ \). \( E_c \) point is the center point formed by the intersection of each bisector \( \angle C \) and \( \angle CDK \). And \( E_d \) point is the center point formed by the intersection of each bisector \( \angle A \) and \( \angle CDK \). \( \rho_a, \rho_b, \rho_c \) and \( \rho_d \) are the lengths of the radii of the circle, each centered on \( E_a, E_b, E_c \) and \( E_d \) and offend the extension side of \( a, b, c \) and \( d \). So will apply

\[
\frac{\rho_a \rho_b}{R_a R_b} = -\tan \frac{A}{2} \tan \frac{B}{2} \tan \frac{C}{2} \tan \frac{D}{2}.
\]
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As well as
\[
\frac{\rho_a \rho_d}{R_c R_d} = -\tan \frac{A + D}{2} \tan \frac{D + C}{2} \tan \frac{A}{2} \tan \frac{C}{2}.
\]

And since it also applies
\[
\tan \frac{A + B}{2} \tan \frac{B + C}{2} = \tan \frac{A + D}{2} \tan \frac{C + D}{2},
\]
then we obtain
\[
\frac{\rho_a \rho_b}{R_a R_b} = \frac{\rho_c \rho_d}{R_c R_d}.
\]

Since \( R_a R_b = R_c R_d \), it should \( \rho_a \rho_b = \rho_c \rho_d \). From the definition of \( \rho_a, \rho_b, \rho_c \) and \( \rho_d \), then must \( \rho_a = \rho_b = \rho_c = \rho_d \). Since the lengths of radii are same, it must be the center point of the circle that is \( E_a = E_b = E_c = E_d \) that are in front of point \( C \), in other words \( \square ABCD \) has excircle.

References

[1] A Gultierrez, Go Geometri, hal 1
http://www.gogeometry.com/problem/p569_quadrilateral_excircles_tangency_point_congruence.htm. accessed 4 August 2015. 08.35 pm


