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SOME ALTERNATIVE CONCETS FOR FUZZY NORMED SPACE AND FUZZY 2-NORMED SPACE

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Abstract

We study some alternative concepts for a fuzzy normed space and a fuzzy 2-normed space together with a fuzzy inner product space and a fuzzy 2-inner product space. The method is to make use of fuzzy points. We also discuss the relation between a normed space and a fuzzy normed space and the generalization of a fuzzy normed space to a fuzzy 2-normed space. Conversely, a fuzzy 2-normed space is reduced to a fuzzy normed space. The same method is applied to fuzzy inner product spaces.

1. Introduction

The theory of fuzzy sets was first introduced by L. Zadeh in 1965. Since then the applications of the fuzzy sets have advanced in many disciplines. In mathematics, the developments of the fuzzy sets have advanced not only in analysis, but also in numerical analysis, operations research, statistics, and algebra [3, 5, 11, 17, 18, 23].

In mathematical analysis, the concept of fuzzy has developed to a fuzzy n-metric space and a fuzzy n- normed space. The concept of fuzzy normed space and fuzzy 2-normed space to fuzzy n- normed space are approached from two sides.

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First, the fuzzy normed space is intuitionisticly approached using the concept of the continue n- norm and the continue n- conorm [3, 21, 22, 23, 26]. On the one hand, the definition of the fuzzy normed space begins by defining a function $N : X \times R \to [0, 1]$, where X is a real linear space [4, 9, 25], on the other hand [14, 15, 22] introduce the fuzzy normed space using the fuzzy point approach of a fuzzy set. However, using the approach they introduce, the previous authors have not developed yet the fuzzy normed space to fuzzy 2-normed space. They also have not developed yet a fuzzy inner product space to a fuzzy 2-inner product space. Then, Mashadi [18] approaches a fuzzy normed space by t-norms and t-conorms, but he does not provide the generalization of a fuzzy 2-normed space.

Based on the above explanation, by a bit modification on the definition of the fuzzy normed space and the fuzzy inner product space in [14, 15, 16], we construct a fuzzy 2-normed space and a fuzzy 2-inner product space.

2. Fuzzy Normed Space and Fuzzy Inner Product Space

Let X be any set. Then a fuzzy set $\tilde{A}$ in X is characterized by a membership function $\mu_{\tilde{A}}(x) : X \to [0, 1]$. Then $\tilde{A}$ can be written as

$$\tilde{A} = \{(x, \mu_{\tilde{A}}(a)) | x \in X, 0 \leq \mu_{\tilde{A}}(x) \leq 1\}.$$

Definition 2.1. A fuzzy point P_x in X is a *fuzzy set* whose membership function is

$$\mu_{P_x}(y) = \begin{cases} \alpha, & \text{if } y = x, \\ 0, & \text{else} \end{cases}$$

for all $y \in X$, where $0 < \alpha < 1$. We denote fuzzy points as x_α or (x, α).

Let **X** be a vector space over a field K, and let $\tilde{\mathbf{A}}$ be a fuzzy set in **X**. Then $\tilde{\mathbf{A}}$ is said to be a *fuzzy subspace* in **X** if for all $x, y \in \mathbf{X}$ and $\lambda \in \mathbf{K}$ satisfy

i. $\mu_{\tilde{A}}(x + y) \geq \min\{\mu_{\tilde{A}}(x), \mu_{\tilde{A}}(y)\}$.

ii. $\mu_{\tilde{A}}(\lambda x) \geq \mu_{\tilde{A}}(x)$.

Definition 2.2. Let x_a be a *fuzzy point* and $\tilde{\mathbf{A}}$ be a *fuzzy set* in X. Then $x_a \in \tilde{\mathbf{A}}$ if $\alpha \leq \mu_{\tilde{A}}(x)$.

Definition 2.3. Let $\mathbf{X}$ be a vector space over a field K ($K = R$ or $K = C$), fix $I = [0, 1]$. Then a mapping from $\tilde{\mathbf{A}} \times \tilde{\mathbf{A}}$ to a field $\mathbf{K}$, that is for each pair of fuzzy vector x_α, y_β there exists a associated scalar written as (x_α, y_β) or $\langle x, y\rangle(\lambda)$, where $\lambda = \min\{\alpha, \beta\}$ and $\alpha, \beta \in (0, 1]$, for all vectors x_α, y_β, z_γ, where $\lambda = \min\{\alpha, \beta, \gamma\}$ and skalar r holds if it satisfies the following properties:

(FIP1) $\langle x, x\rangle(\lambda) \geq 0$ and $\langle x, x\rangle(\lambda) = 0$ if and only if $x = 0$.

(FIP2) $\langle x, y\rangle(\lambda) = \overline{\langle y, x\rangle}(\lambda)$.

(FIP3) $\langle rx, y\rangle(\lambda) = r\langle x, y\rangle(\lambda)$.

(FIP4) $\langle x + y, z\rangle(\lambda) = \langle x, z\rangle(\lambda) + \langle y, z\rangle(\lambda)$.

(FOP5) If $0 < \sigma \leq \alpha < 1$, then $\langle x, x\rangle(\alpha) \leq \langle x, x\rangle(\sigma)$, then there exists $0 < \alpha_n < \alpha$ so that $\lim_{n\to\infty}\langle x, x\rangle(\alpha_n) = \langle x, x\rangle(\alpha)$.

Then $\langle ., .\rangle(.)$ is a fuzzy inner product and $(\mathbf{X}; \langle ., .\rangle(.))$ is a fuzzy inner product space.

Kamali and Mazaheri [10] show that if $(\mathbf{X}; \langle ., .\rangle)$ is an ordinary inner product space, if we define fuzzy inner product space as $\langle x, y\rangle(\lambda) = \frac{1}{\lambda}\langle x, y\rangle$ for all $x_\alpha, y_\beta \in \tilde{\mathbf{A}}$, where $\lambda = \min\{\alpha, \beta\}$ and $\alpha, \beta \in (0, 1]$, then $(\mathbf{X}; \langle ., .\rangle(.))$ is a fuzzy inner product space. So, if $(\mathbf{X}; \langle ., .\rangle(.))$ is a fuzzy inner product space and we define $\langle x, y\rangle = \langle x, y\rangle(\lambda) = \langle x_\alpha, y_\beta\rangle$, then we have $(\mathbf{X}; \langle ., .\rangle)$ an ordinary inner product space. Kider [14] proves that $(\mathbf{X}; \langle ., .\rangle(.))$ is a fuzzy inner product space, for all $x_\alpha, y_\beta \in \tilde{\mathbf{A}}$, fuzzy Cauchy-Schwarz inequality holds, that is $|\langle x, y\rangle(\lambda)|^2 \leq (x, x)(\lambda) \cdot (y, y)(\lambda)$, where $\alpha, \beta \in (0, 1]$ and $\lambda = \min\{\alpha, \beta\}$.

Definition 2.4. Let $\mathbf{X}$ be a vector space over a field K, and let $\|.\|_f : \tilde{\mathbf{A}}. \to [0, \infty)$ be a function that associates each point x_α in $\mathbf{X}$, $\alpha \in (0, 1]$ nonnegative real numbers $\|.\|_f$ so that

(FN1) $\| x_\alpha \|_f = 0$ if and only if $x = 0$.

(FN2) $\| \lambda x_\alpha \|_f = |\lambda| \| x_\alpha \|,$ for all $\lambda \in K.$

(FN3) $\| x_\alpha + y_\beta \|_f \leq \| x_\alpha \|_f + \| y_\beta \|_f.$

(FN4) If $0 < \sigma \leq \alpha < 1,$ then $\| x_\alpha \|_f \leq \| x_\sigma \|_f,$ and there exists $0 < \alpha_n < \alpha$ such that $\lim_{n\to\infty} \| x_{\alpha_n} \|_f = \| x_\alpha \|.$

Then $\| . \|_f$ is called a *fuzzy norm* and $(\mathbf{X} \| . \|_f)$ is called a *fuzzy normed space*.

Definition 2.4 above is slightly different from the definition given by Kider [14], where Kider gives the definition in the form of

(FN1′) If $\alpha = 0,$ then $\| x_\alpha \|_f = 0.$

(FN1″) If $\alpha \neq 0,$ $\| x_\alpha \|_f = 0$ if and only if $x = 0.$

However, property (FN1′) should not be exist, because in the previous it mentioned that $\alpha \in (0, 1].$ So, for the next discussion we remain using Definition 2.4 for the definition of a fuzzy norm.

Based on Definition 2.4, it shows that in [15-17] the relationship between an ordinary norm and a fuzzy norm that is by defining $\| x_\alpha \|_f = \frac{1}{\alpha} \| x \|$ for all $x_\alpha \in \mathbf{X},$ where $\alpha \in (0, 1],$ then $(\mathbf{X}; \| . \|_f)$ is a fuzzy normed space. Conversely if $\| x_\alpha \|_f$ is a fuzzy normed space, then it is defined that $\| x \| = \| x_1 \|_f = \| (x, 1) \|_f$ is a normed space. So, if we have a normed space, then we can always construct a fuzzy normed space and vice versa, in other words a normed space is equivalent to a fuzzy normed space. So, all properties that hold for a norm space also hold for a fuzzy normed space.

Kider [14] and Kider [15] mention that if $(\mathbf{X}; \langle ., . \rangle(.))$ is a fuzzy inner product space, then the fuzzy norm $\| x_\alpha \| = [\langle x_\alpha, x_\alpha \rangle]^{\frac{1}{2}}$ satisfies a fuzzy parallelogram, that is,

$$\| x + y \|^2(\lambda) + \| x - y \|^2(\lambda) = 2[\| x \|^2(\lambda) + \| y \|^2(\lambda)]$$

for each $x_\alpha - y_\beta \in P(X),$ where $\alpha, \beta \in (0, 1]$ and $\lambda = \min\{\alpha, \beta\}.$

3. Fuzzy 2-inner Product Space and Fuzzy 2-normed Space

Referring to above definitions, for what follows we define a fuzzy 2-inner product space and fuzzy 2-normed space.

Definition 3.1. A fuzzy 2-inner product space on $\mathbf{X}$, where $\mathbf{X}$ is a vector space over field K, is a mapping from $\tilde{\mathbf{A}} \times \tilde{\mathbf{A}} \times \tilde{\mathbf{A}}$ into field K, that is for each pair of vectors $x_\alpha, y_\beta, z_\gamma$ there exists an associated scalar $\langle x_\alpha, y_\beta | z_\gamma \rangle$ or $\langle x, y | z \rangle(\lambda)$, where $\lambda = \min\{\alpha, \beta, \gamma\}$, $\alpha, \beta, \gamma \in (0, 1]$ and is called a *fuzzy* 2-*inner product* of x_α, y_β and z_γ so that for all fuzzy vectors x_α, y_β, z_γ, w_δ, where $\lambda = \min\{\alpha, \beta, \gamma, \delta\}$ and scalar *i* if it satisfies the following properties:

(F2IP1) $\langle x, x | w \rangle(\lambda) \geq 0$, for each $x_\alpha, w_\delta \in P(X)$,

$\langle x, x | w \rangle(\lambda) = 0$ if and only if x_α, w_δ are linearly dependent.

(F2IP2) $\langle x, x | w \rangle(\lambda) = \langle \overline{w, w | x} \rangle(\lambda)$.

(F2IP3) $\langle x, y | w \rangle(\lambda) = \langle \overline{y, x | w} \rangle(\lambda)$.

(F2IP4) $\langle rx, y | w \rangle(\lambda) = r\langle x, y | w \rangle(\lambda)$.

(F2IP5) $\langle x + y, z | w \rangle(\lambda) = \langle x, z | w \rangle(\lambda) + \langle y, z | w \rangle(\lambda)$.

(F2IP6) If $0 < \sigma \leq \rho < 1$, then $\langle x, x | w \rangle(\rho) \leq \langle x, x | w \rangle(\sigma)$ and there exist $0 < \rho_n < \rho$ so that $\lim_{n\to\infty} \langle x, x | w \rangle(\rho_n) = \langle x, x | w \rangle(\rho)$.

If all of the six properties are satisfied, then $(\mathbf{X}; \langle ., .|. \rangle(.))$ is called a *fuzzy* 2-*inner product space*.

Let $(\mathbf{X}; \langle ., .|. \rangle(.))$ be an ordinary inner product space. Define $\langle x, y | w \rangle(\lambda) = \frac{1}{\lambda}\langle x, y | z \rangle$, then

1. It is clear that $\langle x, x | w \rangle(\lambda) = \frac{1}{\lambda}\langle x, x | z \rangle \geq 0$.

Let $\langle x, x | w \rangle(\lambda) = 0 \Leftrightarrow \langle x, x | z \rangle = 0 \Leftrightarrow x, z$ be linearly dependent.

2. $\langle x, x | w \rangle (\lambda) = \frac{1}{\lambda} \langle x, x | w \rangle = \frac{1}{\lambda} \langle w, w | x \rangle = \frac{1}{\lambda} \overline{\langle w, w | x \rangle} = \overline{\langle w, w | x \rangle} (\lambda).$

3. $\langle x, y | w \rangle (\lambda) = \frac{1}{\lambda} \langle x, y | w \rangle = \frac{1}{\lambda} \overline{\langle y, x | w \rangle} = \overline{\langle y, x | w \rangle} (\lambda).$

4. $\langle rx, y | w \rangle (\lambda) = \frac{1}{\lambda} \langle rx, y | w \rangle = r \cdot \frac{1}{\lambda} (\langle x, y | w \rangle = r \langle x, y | w \rangle) (\lambda).$

5.

$$\begin{aligned} \langle x + y, z | w \rangle (\lambda) &= \frac{1}{\lambda} \langle x + y, z | w \rangle \\ &= \frac{1}{\lambda} (\langle x, z | w \rangle + \langle y, z | w \rangle) \\ &= \frac{1}{\lambda} \langle x, z | w \rangle + \frac{1}{\lambda} \langle y, z | w \rangle \\ &= \langle x, z | w \rangle (\lambda) + \langle y, z | w \rangle (\lambda). \end{aligned}$$

6. If $0 < \sigma \leq \rho < 1$, then $\frac{1}{\rho} \leq \frac{1}{\sigma}$, then $\frac{1}{\sigma} \langle x, x | w \rangle = \frac{1}{\sigma} \langle x, x | w \rangle$ or $\langle x, x | w \rangle (\rho) \leq \langle x, x | w \rangle (\sigma).$

Next, let $\rho_n = \left(1 - \frac{1}{n}\right) \rho$ so that

$$\begin{aligned} \lim_{n \to \infty} \langle x, x | w \rangle (\rho_n) &= \lim_{n \to \infty} \frac{1}{\rho_n} \langle x, x | w \rangle \\ &= \frac{1}{\rho} \langle x, x | w \rangle \\ &= \langle x, x | w \rangle (\rho). \end{aligned}$$

So, from the six above properties satisfied, it can be concluded that for every $(\mathbf{X}; \langle ., . \rangle (.))$ we can always construct a fuzzy inner product space $(\mathbf{X}; \langle ., .|. \rangle (.))$.

Proposition 3.2. *Let* $(\mathbf{X}; \langle ., . \rangle)$ *be a fuzzy inner product space. Define a function*

$$\langle x, y | w \rangle_s (\lambda) = \begin{vmatrix} \langle x, y \rangle (\lambda) & \langle x, w \rangle (\lambda) \\ \langle w, y \rangle (\lambda) & \langle w, w \rangle (\lambda) \end{vmatrix}.$$

It can be showed that $\langle x, y | w \rangle_s (\lambda)$ *is a fuzzy* 2*-inner product*.

Proof. (F2-IP1) By Cauchy-Schwarz inequalities, it is clear that $\langle x, x | w \rangle (\lambda) \geq 0$, for all $x_\alpha, w_\delta \in P(X)$.

(F2-IP2)

$$\langle rx, y | w \rangle (\lambda) = \begin{vmatrix} \langle rx, y \rangle (\lambda) & \langle rx, w \rangle (\lambda) \\ \langle w, y \rangle (\lambda) & \langle w, w \rangle (\lambda) \end{vmatrix}$$

$$= r \begin{vmatrix} \langle x, y \rangle (\lambda) & \langle x, w \rangle (\lambda) \\ \langle w, y \rangle (\lambda) & \langle w, w \rangle (\lambda) \end{vmatrix}$$

$$= r \langle x, y | w \rangle (\lambda).$$

(F2-IP3) Thus for property (F2IP4)

$$\langle x, y | w \rangle_s (\lambda) = \begin{vmatrix} \langle x, y \rangle (\lambda) & \langle x, w \rangle (\lambda) \\ \langle w, y \rangle (\lambda) & \langle w, w \rangle (\lambda) \end{vmatrix}$$

$$= \begin{vmatrix} \langle \overline{y, x} \rangle (\lambda) & \langle \overline{w, x} \rangle (\lambda) \\ \langle \overline{y, w} \rangle (\lambda) & \langle w, w \rangle (\lambda) \end{vmatrix}$$

$$= \langle \overline{y, x | w} \rangle_s (\lambda).$$

(F2-IP4) Next, note that

$$\langle x, x | w \rangle_s (\lambda) = \begin{vmatrix} \langle x, x \rangle (\lambda) & \langle x, w \rangle (\lambda) \\ \langle w, x \rangle (\lambda) & \langle w, w \rangle (\lambda) \end{vmatrix}$$

$$= \begin{vmatrix} \langle w, w \rangle (\lambda) & \langle w, x \rangle (\lambda) \\ \langle x, w \rangle (\lambda) & \langle x, x \rangle (\lambda) \end{vmatrix}$$

$$= \langle w, w | x \rangle_s (\lambda). \qquad (*)$$

(F2-IP5) Note the relationship

$$\langle x + y, z | w \rangle (\lambda) = \begin{vmatrix} \langle x + y, z \rangle (\lambda) & \langle x + y, w \rangle (\lambda) \\ \langle w, y \rangle (\lambda) & \langle w, w \rangle (\lambda) \end{vmatrix}$$

$$= \begin{vmatrix} \langle x, z \rangle(\lambda) + \langle y, z \rangle(\lambda) & \langle x, w \rangle(\lambda) + \langle y, w \rangle(\lambda) \\ \langle w, y \rangle(\lambda) & \langle w, w \rangle(\lambda) \end{vmatrix}$$

$$= \begin{vmatrix} \langle x, z \rangle(\lambda) & \langle x, w \rangle(\lambda) \\ \langle w, y \rangle(\lambda) & \langle w, w \rangle(\lambda) \end{vmatrix} + \begin{vmatrix} \langle y, z \rangle(\lambda) & \langle y, w \rangle(\lambda) \\ \langle w, y \rangle(\lambda) & \langle w, w \rangle(\lambda) \end{vmatrix}$$

$$= \langle x, z \mid w \rangle(\lambda) + \langle y, z \mid w \rangle(\lambda).$$

(F2-IP6) Let $0 < \sigma \le \alpha < 1$. Then $\frac{1}{\alpha} \le \frac{1}{\sigma}$ and so

$$\frac{1}{\alpha}\langle x, x \mid w \rangle \le \frac{1}{\alpha}\langle x, x \mid w \rangle \quad \text{or} \quad \langle x, x \mid w \rangle(\alpha) \le \langle x, x \mid w \rangle(\sigma). \qquad \blacksquare$$

Remark

1. Notice the relationship

$$\langle x, y \mid w \rangle_s(\lambda) = \begin{vmatrix} \langle x, y \rangle(\lambda) & \langle x, w \rangle(\lambda) \\ \langle w, y \rangle(\lambda) & \langle w, w \rangle(\lambda) \end{vmatrix}$$

$$= \begin{vmatrix} \langle \overline{y, x} \rangle(\lambda) & \langle \overline{w, x} \rangle(\lambda) \\ \langle \overline{y, w} \rangle(\lambda) & \langle \overline{w, w} \rangle(\lambda) \end{vmatrix},$$

$$\begin{vmatrix} \langle \overline{y, x} \rangle(\lambda) & \langle \overline{y, w} \rangle(\lambda) \\ \langle \overline{w, x} \rangle(\lambda) & \langle \overline{w, w} \rangle(\lambda) \end{vmatrix} = \langle \overline{y, x \mid w} \rangle_s(\lambda)$$

but

$$\langle y, x \mid w \rangle_s(\lambda) = \begin{vmatrix} \langle y, x \rangle(\lambda) & \langle y, w \rangle(\lambda) \\ \langle w, x \rangle(\lambda) & \langle w, w \rangle(\lambda) \end{vmatrix}$$

$$\ne \langle x, y \mid w \rangle_s(\lambda).$$

So $\langle x, y \mid w \rangle_s(\lambda) = \langle \overline{y, x \mid w} \rangle_s(\lambda) \ne \langle x, y \mid w \rangle_s(\lambda)$.

2. Notice the relationship

$$\langle \overline{w, w \mid x} \rangle_s(\lambda) = \begin{vmatrix} \langle \overline{w, w} \rangle(\lambda) & \langle \overline{w, x} \rangle(\lambda) \\ \langle \overline{x, w} \rangle(\lambda) & \langle \overline{x, x} \rangle(\lambda) \end{vmatrix}$$

$$= \begin{vmatrix} \overline{\langle w, w \rangle}(\lambda) & \overline{\langle x, w \rangle}(\lambda) \\ \overline{\langle w, x \rangle}(\lambda) & \overline{\langle x, x \rangle}(\lambda) \end{vmatrix}$$

$$= \begin{vmatrix} \overline{\langle w, w \rangle}(\lambda) & \overline{\langle w, x \rangle}(\lambda) \\ \overline{\langle x, w \rangle}(\lambda) & \overline{\langle x, x \rangle}(\lambda) \end{vmatrix}$$

$$= \langle w, w | x \rangle_s (\lambda). \qquad (**)$$

So, from (*) and (**) it states that

$$\langle x, x | w \rangle_s (\lambda) = \overline{\langle w, w | x \rangle}_s (\lambda)$$

$$= \langle w, w | x \rangle_s (\lambda).$$

Then the properties (F2IP3) above can be replaced by $\langle x, x|w \rangle_s (\lambda) = \langle w, w|x \rangle_s (\lambda)$.

Referring to fuzzy normed space in [14, 15], if we define for every fuzzy point $x_\alpha + y_\beta = (x + y)_\lambda$ by $\lambda = \max\{\alpha, \beta\}$, then we can define a fuzzy 2-normed space as follows:

Definition 3.3. Let **X** be a vector space over a field K, and let $\| ., . \|_f : \tilde{\mathbf{A}} \to [0, \infty)$ be a function associated every point x_α, y_β in $\tilde{\mathbf{A}}$, $\alpha, \beta \in (0, 1]$ nonnegative real numbers $\| ., . \|_f$ so that

(FN1) $\| x_\alpha, y_\beta \|_f = 0$ if and only if x and y are linearly dependent.

(FN2) $\| x_\alpha, y_\beta \|_f = \| y_\beta, y_\alpha \|_f$ for all x_α, y_β in X.

(FN3) $\| \lambda x_\alpha, y_\beta \|_f = | \lambda | \| x_\alpha, y_\beta \|$, for all x_α, y_β in X and any $\lambda \in K$.

(FN4) $\| x_\alpha + y_\beta, z_\gamma \|_f \le \| x_\alpha, z_\gamma \|_f + \| y, z_\gamma \|_f$ for all x_α, y_β and z_γ in **X**.

(FN5) If $0 < \rho \le \alpha < 1$ and $0 < \alpha \le \beta < 1$, then $\| x_\alpha, y_\beta \|_f < \| x_\rho, y_\sigma \|_f$, and there exists $0 < \alpha_n < \alpha$ and $0 < \beta_n < \beta$ so that $\lim_{n \to \sim} \| x_{\alpha_n}, y_{\beta_n} \|_f = \| x_\alpha, y_\beta \|_f$.

Then $\| ., . \|_f$ is called a *fuzzy* 2-*norm* and $(\mathbf{X}, \| ., . \|_f)$ is called a *fuzzy* 2-*normed* *space*. By Definition 3.3 above, we then have the relationship between the ordinary norm and fuzzy norm as follows:

Proposition 3.4. *Let* $(\mathbf{X}\|\,.,\,.\,\|)$ *be an ordinary* 2-*normed space. Define* $\| x_\alpha, y_\beta \|_f = \frac{1}{\delta}\| x, y \|,$ *where* $\delta = \max\{\alpha, \beta\},$ *for all* $x_\alpha, y_\beta, z_\gamma \in \tilde{\mathbf{A}}$ *and* $\alpha, \beta, \gamma \in (0, 1],$ *then* $(\mathbf{X}\|\,.,\,.\,\|_f)$ *is a fuzzy* 2-*normed space*.

Proof. Let $x_\alpha, y_\beta \in \tilde{\mathbf{A}},$ where $\alpha, \beta \in (0, 1]$ and $\lambda \in K$. Then

(FN1) $\| x_\alpha, y_\beta \|_f = 0 \Leftrightarrow \frac{1}{\delta}\| x, y \| = 0 \Leftrightarrow \| x, y \| = 0 \Leftrightarrow x$ and y linearly dependent.

(FN2) $\| x_\alpha, y_\beta \|_f = \frac{1}{\delta}\| x, y \| = \frac{1}{\delta}\| y, x \| = \| y_\beta, x_\alpha \|_f.$

(FN3) $\| \lambda x_\alpha, y_\beta \|_f = \frac{1}{\delta}\| \lambda x, y \| = \frac{|\lambda|}{\delta}\| x, y \| = |\lambda| \cdot \| x_\alpha, y_\beta \|_f.$

(FN4)

$\| x_\alpha + y_\beta, z_\gamma \|_f \le \| (x + y)_\tau, z_\beta \|_f,$ where $\tau = \max\{\alpha, \beta\}$

$$= \frac{1}{\rho}\| x + y, z \|$$

$$\le \frac{1}{\rho}\| x, z \| + \frac{1}{\rho}\| y, z \|, \text{ where } \rho = \max\{\tau, \gamma\}$$

$$\le \frac{1}{\mu}\| x, z \| + \frac{1}{\vartheta}\| y, z \|, \text{ where } \mu = \max\{\alpha, \gamma\} \text{ and } \varphi = \max\{\beta, \gamma\}$$

$$= \| x_\alpha, z_\gamma \|_f + \| y_\beta, z_\gamma \|_f.$$

(FN5) If $0 < \rho \le \alpha < 1$ and $0 < \sigma \le \beta < 1,$ let $\delta = \max\{0, \beta\}$ and $\tau = \max\{\rho, \sigma\},$ then $\tau \le \delta,$ so $\frac{1}{\delta} \le \frac{1}{\tau},$ then $\frac{\| x, y \|}{\delta} \le \frac{\| x, y \|}{\tau},$ it means that $\| x, y \|(\delta) \le \| x, y \|(\sigma),$ in other words $\| x_\alpha, y_\beta \|_f < \| x_\rho, y_\sigma \|_f.$ Next let

$\alpha_n = \left(1 - \frac{1}{n}\right)\alpha$ and $\beta_n = \left(1 - \frac{1}{n}\right)\beta$. Then $\lim_{n\to\infty} \| x_{\alpha_n}, y_{\beta_n} \|_f = \lim_{n\to\infty} \frac{1}{\delta_n} \| x, y \|$

$= \frac{1}{\delta} \| x, y \| = \| x_\alpha, y_\beta \|_f$, where $\delta_n = \max\{\alpha_n, \beta_n\}$. ■

Conversely, by Definition 3.3 if $\| x_\alpha, y_\beta \|_f$ is a fuzzy 2-normed space, define $\| x, y \| = \| x_1, y_\beta \|_f$ or $\| x, y \| = \| x_\alpha, y_1 \|_f$, is a 2-normed space. So by Definition 3.3 and Proposition 3.2 it follows that whenever we have a 2-normed space, then we can always construct a fuzzy 2-normed space and vice versa, in other words the 2-normed space equivalent to a fuzzy 2-normed space. So, all properties applied to norm space are also applied to fuzzy normed space.

Theorem 3.5. *Let* $(\mathbf{X}, \| \cdot \|_f)$ *be a fuzzy normed space. Define*

$$\| x_\alpha, y_\beta \|_f = \begin{cases} 0, & \textit{if } \| x_\alpha - y_\beta \|_f = 0, \\ \| x_\alpha \|_f \cdot \| y_\beta \|_f, & \textit{lainnya}. \end{cases}$$

Then $(\mathbf{X}, \| \cdot, \cdot \|_f)$ *is a fuzzy 2-normed space.*

Proof.

(FN.1) It is clear from definition, $\| x_\alpha, y_\beta \|_f = 0$, if and only if x_α, y_β are linearly dependent.

(FN.2) If x_α, y_β are linearly dependent, clearly $\| x_\alpha, y_\beta \|_f = \| y_\beta, x_\alpha \|_f$, then suppose that they are not linearly dependent. Then

$$\begin{aligned} \| x_\alpha, y_\beta \|_f &= \| x_\alpha \|_f \cdot \| y_\beta \|_f \\ &= \| y_\beta \|_f \cdot \| x_\alpha \|_{f^*} \\ &= \| y_\beta, x_\alpha \|_f. \end{aligned}$$

(FN.3) $\| \lambda x_\alpha, y_\beta \|_f = \| \lambda x_\alpha \|_f \cdot \| y_\beta \|_f = |\lambda| \cdot \| x_\alpha \|_f \cdot \| y_\beta \|_f = |\lambda| \| x_\alpha, y_\beta \|_f$.

(FN.4)

$\| x_\alpha + y_\beta, z_\gamma \|_f = \| x_\alpha + y_\beta \|_f \cdot \| z_\gamma \|_f$

$$\leq (\| x_\alpha \|_f + \| y_\beta \|_f) \cdot \| z_\gamma \|_f$$

$$= \| x_\alpha \|_f \cdot \| z_\gamma \|_f + \| y_\beta \|_f \cdot \| z_\gamma \|_f$$

$$= \| x_\alpha, z_\gamma \|_f + \| y_\beta, z_\gamma \|_f.$$

(FN.5) If $\| x_\alpha - y_\beta \|_f = 0$ clearly $\| x_\alpha, y_\beta \|_f < \| x_\rho, y_\alpha \|_f$, then let $\| x_\alpha - y_\beta \|_f \neq 0$, and $\| x_\alpha, y_\beta \|_f = \| x_\alpha \|_f \cdot \| y_\beta \|_f$ so if $0 < \rho \leq \alpha < 1$ and $0 < \sigma \leq \beta < 1$, then $\frac{1}{\alpha} \leq \frac{1}{\rho}$ and $\frac{1}{\beta} \leq \frac{1}{\sigma}$, so $\frac{1}{\alpha}\| x \| \leq \rho\| x \|$ in other words $\| x_\alpha \|_f \leq \| x_\rho \|_f$. Similarly we have $\| x_\beta \|_f \leq \| x_\alpha \|_f$, so $\| x_\alpha \|_f \cdot \| y_\beta \|_f \leq \| x_\rho \|_f \cdot \| y_\sigma \|_f$ that implies that $\| x_\alpha, y_\beta \|_f < \| x_\rho, y_\sigma \|_f$. Next let $\alpha_n = \left(1 - \frac{1}{n}\right)\alpha$ and $\beta_n = \left(1 - \frac{1}{n}\right)\beta$. Then

$$\lim_{n\to\sim} \| x_{\alpha_n}, y_{\beta_n} \|_f = \lim_{n\to\sim} \| x_{\alpha_n} \|_f \cdot \| y_{\beta_n} \|_f$$

$$= \lim_{n\to\sim} \frac{1}{\alpha_n}\| x \| \cdot \frac{1}{\beta_n}\| x \|$$

$$= \frac{1}{\alpha}\| x \| \cdot \frac{1}{\beta}\| x \|$$

$$= \| x_\alpha \|_f \cdot \| y_\beta \|_f$$

$$= \| x_\alpha, y_\beta \|_f.$$

So, $\| x_\alpha, y_\beta \|_f$ is a fuzzy 2-norm over $\mathbf{X}$ and $(\mathbf{X} \| ., . \|_f)$ is a fuzzy 2-normed space. ■

Theorem 3.6. *Let* $(\mathbf{X} \| ., . \|_f)$ *be a fuzzy* 2*-normed space. Define*

$$\| x_\delta \|_f = \| x_\delta, a_\alpha \|_f + \| x_\delta, b_\beta \|_f,$$

where a_α *and* $b_\beta \in \mathbf{X}$ *are two linearly independent vectors. Then* $\| x_\delta \|_f$ *is a fuzzy norm and* $(\mathbf{X} \| . \|_f)$ *is a fuzzy* 2*-normed space.*

Proof.

(FN1) If $x_\delta = 0,$ clearly $\| x_\delta \|_f = 0,$ conversely let $\| x_\delta \|_f = 0.$ Then

$$\| x_\delta, a_\alpha \|_f = \| x_\delta, b_\beta \|_f$$
$$= 0.$$

Then by Definition 3.3, then x_δ with a_α and b_β are linearly independent, so there exist $s, t \in R$ so that $x_\delta = s \cdot a_\alpha = t \cdot b_\beta.$ However, a_α and b_β are also linearly independent, then there must be $\alpha = \beta = 0,$ so that $x_\delta = 0.$

(FN2) It is clear from definitions $\| \lambda x_\delta \|_f = | \lambda | \cdot \| x_\delta \|_f .$

(FN3)

$$\begin{aligned} \| x_\delta + y_\gamma \|_f &= \| x_\delta + y_\gamma, a_\alpha \|_f + \| x_\delta + y_\gamma, b_\beta \|_f \\ &\le \| x_\delta, a_\alpha \|_f + \| y_\gamma, a_\alpha \|_f + \| x_\delta, b_\beta \|_f + \| y_\gamma, b_\beta \|_f \\ &= \| x_\delta \|_f + \| y_\gamma \|_f . \end{aligned}$$

(FN4) If $0 < \sigma \le \delta < 1$ above has been shown, there must hold

$$\| x_\delta, a_\alpha \|_f \le \| x_\alpha, a_\alpha \| \quad \text{and} \quad \| x_\delta, b_\beta \|_f \le \| x_\sigma, b_\beta \|_f$$

that implies

$$\begin{aligned} \| x_\delta \|_f &= \| x_\delta, a_\alpha \|_f + \| x_\delta, b_\beta \|_f \\ &\le \| x_\sigma, a_\alpha \|_f \cdot \| x_\sigma, b_\beta \|_f \\ &= \| x_\sigma \|_f . \end{aligned}$$

Doing the similar manner to the proof of (FN5) in Theorem 3.5 above, we obtain

$$\lim_{n\to\sim} \| x_{\delta_n}, a_{\alpha_n} \|_f = \| x_\delta, a_\alpha \|_f \quad \text{and} \quad \lim_{n\to\sim} \| x_{\delta_n}, b_{\beta_n} \|_f = \| x_\delta, b_\beta \|_f .$$

So,

$$\lim_{n\to\sim} \| x_{\delta_n} \|_f = \lim_{n\to\sim} (\| x_{\delta_n}, a_{\alpha_n} \|_f + \| x_{\delta_n}, a_{\alpha n} \|_f)$$

$$= \lim_{n\to\sim} \| x_{\delta_n}, a_{\alpha_n} \|_f + \lim_{n\to\sim} \| x_{\delta_n}, b_{\beta_n} \|_f$$

$$= \| x_\delta, a_\alpha \|_f + \| x_\delta, b_\beta \|_f$$

$$= \| x_\delta \|_f. \qquad \blacksquare$$

Next, it will be shown another alternative to construct a fuzzy normed space from a fuzzy 2-normed space.

Theorem 3.7. *Let* $(\mathbf{X}, \| ., . \|_f)$ *be a fuzzy* 2*-normed space. Define*

$$\| x_\delta \|_{1f}^* = [\| x_\delta, a_\alpha \|_f^2 + \| x_\delta, b_\beta \|_f^2]^{\frac{1}{2}},$$

where a_α *and* $b_\beta \in \mathbf{X}$ *are two linearly independent vectors. Then* $\| x_\delta \|_f$ *is a fuzzy norm and* $(\mathbf{X}, \| . \|_f)$ *is a fuzzy* 2*-normed space.*

Proof.

(FN1) Let $x_\delta = 0$. Then it is clear that $\| x_\delta \|_{1f}^* = 0$. Conversely, let $\| x_\delta \|_{1f}^* = 0$, it will be shown that $x_\delta = 0$. Then, there must $\| x_\delta, a_\alpha \|_f^2 = 0$ and $\| x_\delta, b_\beta \|_f^2 = 0$, if and only if $x_\delta = k_1 \cdot a_\alpha = k_2 \cdot b_\beta$, for some $k_1, k_2 \in R$, since a_α and b_β there must $k_1 = k_2 = 0$, so $X_\delta = 0$.

(FN2)

$$\| rx_\delta \|_{1f}^* = [\| rx_\delta, a_\alpha \|_f^2 + \| rx_\delta, b_\beta \|_f^2]^{\frac{1}{2}}$$

$$= [r^2(\| x_\delta, a_\alpha \|_f^2 + \| x_\delta, b_\beta \|_f^2]^{\frac{1}{2}}$$

$$= r[\| x_\delta, a_\alpha \|_f^2 + \| x_\delta, b_\beta \|_f^2]^{\frac{1}{2}}$$

$$= r \cdot \| x_\delta \|_{1f}^*.$$

(FN3)

$$\| x_\delta + y_\rho \|_{1f}^* = [\| x_\delta + y_\rho, a_\alpha \|_f^2 + \| x_\delta + y_\rho, b_\beta \|_f^2]^{\frac{1}{2}},$$

$$(\| x_\delta + y_\rho \|_{1f}^*)^2 = [\| x_\delta + y_\rho, a_\alpha \|_f^2 + \| x_\delta + y_\rho, b_\beta \|_f^2]$$

$$\leq [(\| x_\delta, a_\alpha \|_f + \| y_\rho, a_\alpha \|_f)^2 + (\| x, b_\beta \|_f + \| y_\rho, b_\beta \|_f)^2]$$

$$\leq [\| x_\delta, a_\alpha \|_f^2 + \| y_\rho, a_\alpha \|_f)_f^2 + \| x_\delta, b_\beta \|_f^2 + \| y_\rho, b_\beta \|_f^2]$$

$$\leq [\| x_\delta, a_\alpha \|_f^2 + \| x_\delta, b_\beta \|_f^2 + \| y_\rho, a_\alpha \|_f^2 + \| y_\rho, b_\beta \|_f^2]$$

$$\leq [\| x_\alpha, a_\alpha \|_f^2 + \| x_\alpha, b_\beta \|_f^2]^{\frac{1}{2}} + [\| y_\rho, a_\alpha \|_f^2 + \| y_\rho, b_\beta \|_f^2]^{\frac{1}{2}}.$$

So,

$$\| x_\delta + y_\rho \|_{1f}^* \leq \| x_\delta \|_{1f}^* + \| y_\rho \|_{1f}^*.$$

(FN4) If $0 < \sigma \leq \delta < 1$, since $(\mathbf{X}, \| ., . \|_f)$ is a fuzzy 2-normed space, then it holds $\| x_\delta, a_\alpha \|_f \leq \| x_\sigma, a_\alpha \|_f$ and $\| x_\delta, y_\beta \|_f \leq \| x_\sigma, y_\beta \|_f$ such that

$$\| x_\delta \|_{1f}^* = [\| x_\delta, a_\alpha \|_f^2 + \| x_\delta, b_\beta \|_f^2]^{\frac{1}{2}}$$

$$= [\| x_\delta, a_\sigma \|_f^2 + \| x_\delta, b_\beta \|_f^2]^{\frac{1}{2}}$$

$$= \| x_\sigma \|_{1f}^{*2}.$$

Next, doing the similar proof to (FN5) Theorem 3.6 above, we have $\lim_{n\to\sim} \| x_{\delta_n} \|_{1f}^* = \| x_\delta \|_{1f}^*$.

From the above four conditions it implies that

$$\| x_\delta \|_{1f}^* = [\| x_\delta, a_\alpha \|_f^2 + \| x_\delta, b_\beta \|_f^2]^{\frac{1}{2}}$$

is a fuzzy normed space. ■

4. Orthogonality

On a norm space there are many types of orthogonalities that are introduced by many authors. However, in this paper we discuss only three orthogonalities as follows:

Definition 4.1. Let $(\mathbf{X}; \|.\|)$ be a norm space. Then

i. Vector x is said to be a *P-orthogonal* to vector y if $\| x + y \|^2 = \| x \|^2 + \| y \|^2$.

ii. Vector x is said to be an *I-orthogonal* to vector y if $\| x + y \|^2 \geq \| x - y \|^2$.

iii. Vector x is said to be a *B-orthogonal* to vector y if $\| x + \propto y \|^2 \geq \| x \|^2$ for each $\alpha \in R$.

Meanwhile the concepts of orthogonality in a 2-norm space have been developed by many authors as the following definition states:

Definition 4.2. Let $(\mathbf{X}; \|., .\|)$ be a norm space. Then

i. Vector x is said to be a *P-orthogonal* to vector y if $\| x + y, z \|^2 = \| x, z \|^2 + \| y, z \|^2$ for each $z \in X$.

ii. Vector x is said to be an *I-orthogonal* to vector y if $\| x + y, z \|^2 \geq \| x - y, z \|^2$.

iii. Vector x is said to be a *B-orthogonal* to vector y if $\| x + \propto y, z \|^2 \geq \| x, z \|^2$ for each $\alpha \in R$ and for each $z \in X$.

Definition 4.2 attracts so many criticisms [4, 7, 8, 10], specially for *B-othogonal*. We also critisize that definition, because if the above definition is used then the three orthogonalities are nothing. Meanwhile if we make use of the definitions of orthogonalities from [1, 2], then one of the vector x or y must be a null vector, the same view is also provided by [4, 10].

Based on the modified definitions of orthogonalities in 2-norm space including the ones given in [7, 8], in what follow we give the modified definitions of orthogonalities in a fuzzy 2-norm space.

Definition 4.3.

i. x_α is said to be *P-orthogonal* to represented y_β by $(x_\alpha \perp_P y_\beta)$ if t here exist subspace $V \subseteq \mathbf{X}$, with codimension 1 such that

$$\| x_\alpha + y_\beta, z_\gamma \| \neq 0 \quad \text{and} \quad \| x_\alpha, z_\gamma \|_f^2 + \| y_\beta, z_\gamma \|_f^2 = \| x_\alpha + y_\beta, z_\gamma \|_f^2$$

$$\text{for all} \quad z_\gamma \in V.$$

ii. x_α is said to be an *I-orthogonal* to y_β represented by $(x_\alpha \perp_I y_\beta)$ if there exists subspace $V \subseteq \mathbf{X}$, with codimension 1 such that

$$\| x_\alpha + y_\beta, z_\gamma \|_f \neq 0 \quad \text{and} \quad \| x_\alpha + y_\beta, z_\gamma \|_f^2 = \| x_\alpha - y_\beta, z_\gamma \|_f^2$$

$$\text{for all} \quad z_\gamma \in V.$$

iii. x_α is said to be *B-orthogonal* to y_β represented by $(x_\alpha \perp_b y_\beta)$ if there exists subspace $V \subseteq \mathbf{X}$, with codimension 1 such that

$$\| x_\alpha, z_\gamma \|_f \neq 0 \quad \text{and} \quad \| x_\alpha, z_\gamma \|_f \leq \| x_\alpha + k y_\beta, z_\gamma \|_f,$$

$$\text{for all} \quad k \in \mathbf{R}, \quad z_\gamma \in V.$$

From Definition 4.3, it remains to agree that in a fuzzy 2-inner product space the three types of the orthogonalities are equivalent to an ordinary orthogonality since for any norm space $(\mathbf{X}; \| . \|)$ we can always construct a fuzzy norm space $(\mathbf{X}; \| . \|_f)$. Then by Theorem 3.5 for any fuzzy norm space we can always construct a fuzzy 2-normed space $(\mathbf{X}. \| ., . \|_f)$. So, the orthogonal properties that hold for an ordinary orthogonal also hold for a fuzzy 2-normed space. However, the interesting case for the analysis is that the relationship for a standard fuzzy 2-normed space, if $\| x_\alpha, y_\beta \| \neq 0$, there always exists $z \notin \text{span}\{x, y\}$ such that $x_\alpha \perp_P y_\beta$. It can be stated in the following theorem.

Theorem 4.4. *Let* $(X. \| ., . \|_f)$ *be a standard fuzzy* 2-*normed space and* $0 \neq x_\alpha, y_\beta \in X$. *If* $\| x_\alpha, y_\beta \| \neq 0$, *then there exists* $z \notin \text{span}\{x, y\}$ *such that* $x_\alpha \perp_P y_\beta$.

Proof. First, select a norm vector w_δ orthogonal to a vector space spanned by

$\| x_\alpha, y_\beta \|$, where $\delta = \min\{\alpha, \beta\}$, then construct vector $z_\delta = x_\alpha \pm y_\beta + \sigma \cdot w_\delta$. Assume that $\| x_\alpha \| = \| y_\beta \| = 1$. Then we obtain $\| z_\delta \|^2 = 2 \pm 2\langle x + y \rangle + \sigma^2$ and also $\langle x_\alpha, y_\beta | z_\delta \rangle = 0$ holds. This shows that $\sigma^2 = \pm \frac{1 - (x_\alpha, y_\beta)^2}{(x_\alpha, y_\beta)}$. Then we can select a value of σ to obtain a value of z_δ satisfying $\langle x_\alpha, y_\beta | z_\delta \rangle = 0$. ■

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