Mashadi, Syamsudhuha, MDH Gamal dan M. Imran, (Eds)
Proceedings of the International Seminar on Mathematics and Its Usage in Other Areas
November 11 – 12, 2010
ISBN. 978-979-1222-95-2

FUZZY ON n-BANACH SPACES

Yuslenita Muda

Department of Mathematics
Faculty of Science and Technology
State Islamic University of Sultan Syarif Kasim
Riau, Pekanbaru, Indonesia
Postcode 28294

Email: ilen_77@hotmail.com or ilen_77@uin-suska.ac.id

Abstract

The main aim of this paper is to introduce the notion of fuzzy of n-Banach space with respect to α-n-norm for it is a complete fuzzy n -normed linear with respect to α- n -norm.

Keywords: Fuzzy n -normed spaces, Fuzzy n -complete normed spaces

1. Introduction

The theory of n-normed space has been introduced by Gunawan and Mashadi. They gave a simple way to derive an $(n-1)$-norm from the n-norm and realized that any n-normed space is an $(n-1)$-normed space. They also showed that, in certain cases, the $(n-1)$-norm can be derived from the n-norm in such a way that the convergence and completeness in the n-norm is equivalent to those in the derived $(n-1)$-norm.

The meanwhile, Surender and Hemen Dutta investigated some properties of linear n-normed spaces and obtained necessary and sufficient conditions for n-norms to be equivalent on linear n-normed spaces. And in [6], Somasundaram and Thangaraj defined the concept of 2-fuzzy 2-normed linear space.

In [1], Al Narayanan and S. Vijayabalaji introduced the notion of fuzzy n-normed linear space as a generalization of n-normed space. In the other side, Elagan, Zayed and Nofal introduced the notion of the absolutely convergent series, and finite convergence sequence in fuzzy n-normed spaces.

Now, I try to introduce the notion of fuzzy on n-Banach spaces which obtained from the property of complete n-normed space.

2. Preliminaries

At the first of this section let we review the fuzzy normed spaces that have been given by Saadati and M. Vaezpour. And several definitions that have been discussed by Narayanan and Vijayabalaji.

Definition 2.1 Let X be a linear space over F (field of real or complex numbers). A fuzzy subset N of $X \times \mathbb{R}$ ($\mathbb{R}$ set of real numbers) is called a fuzzy norm on X if and only if for all $x, u \in X$ and $c \in F$,

(N1) for all $t \in \mathbb{R}$ with $t \leq 0$, $N(x,t) = 0$,
(N2) for all $t \in \mathbb{R}$ with $t > 0$, $N(x,t) = 1$ jika dan hanya jika $x = 0$,
(N3) for all $t \in \mathbb{R}$ with $t > 0$, $N(cx,t) = N\left(x, \frac{t}{|c|}\right)$, jika $c \neq 0$,
(N4) for all $s,t \in \mathbb{R}$, $x,u \in X$, $N(x+u, s+t) \geq \min\{N(x,s), N(u,t)\}$,
(N5) $N(x,\circ)$ is a non decreasing function of $A = \pi r^2$in $\mathbb{R}$ and $\lim_{t\to\infty} N(x,t) = 1$.

The pair (X, N) will be referred to as a fuzzy normed linear space.

Lemma 2.2 Let N be a fuzzy norm. Then

(i) $N(x,t)$ is non decreasing with respect to t for each $x \in X$.

(ii) $N(x-y,t) = N(y-x,t)$

Lemma 2.3 Let (X,N) be a fuzzy normed space. If we define

$$M(x,y,t) = N(x-y,t),$$

Then M is a fuzzy metric on X, which is called the fuzzy metric induced by the fuzzy norm N.

Lemma 2.4 If (X,N) is a fuzzy normed space, then

(a) The function $(x,y) \to x+y$
(b) The function $(\alpha,x) \to \alpha x$ is continous.

Definition 2.5 The fuzzy normed space (X,N) is said to be a *fuzzy banach space* whenever X is complete with respect to the fuzzy metric induced by fuzzy norm.

To complete this paper, let us consider again the definitions that have been given by Gunawan and Mashadi, Bag and Samantha, and Narayanan and Vijayabalaji.

Definition 2.6 Let $n \in \mathbb{N}$ and let X be a real vector space of dimension $d \geq n$ (Here we allow d to be infinite). A real-valued function $\|\cdot,\cdots,\cdot\|$ on X^n satisfiying the following four properties:

i) $\|x_1,\cdots,x_n\| = 0$ if only if $x_1,\cdots,x_n$ are linearly dependent;
ii) $\|x_1,\cdots,x_n\|$ is invariant under permutation;
iii) $\|x_1,\cdots,x_{n-1},\alpha x_n\| = |\alpha|\|x_1,\cdots,x_{n-1},x_n\|$
iv) $\|x_1,\cdots,x_{n-1},y+z\| \leq \|x_1,\cdots,x_{n-1},y\| + \|x_1,\cdots,x_{n-1},z\|$,

is called an n-norm on X and the pair $(X,\|\cdot,\cdots,\cdot\|)$ is called an n-normed space.

Next, we will see the definition of fuzzy n-normed.

Definition 2.7 Let X be a linear space over a real field F. A Fuzzy subset N of $\underbrace{X\times\cdots\times X}_{n}\times\mathbb{R}$ is called a fuzzy n-norm on X if and only if

(N1) for all $t \in \mathbb{R}$ with $t \leq 0$, $N(x_1,x_2,\cdots,x_n,t) = 0$,
(N2) for all $t \in \mathbb{R}$ with $t > 0$, $N(x_1,x_2,\cdots,x_n,t) = 1$ jika dan hanya jika $x_1,x_2,\cdots,x_n$ are linearly dependent,
(N3) $N(x_1,x_2,\cdots,x_n,t)$ is invariant under any permutation of $x_1,x_2,\cdots,x_n$,
(N4) for all $t \in \mathbb{R}$ with $t > 0$,

$$N(x_1,x_2,\cdots,cx_n,t) = N\left(x_1,x_2,\cdots,x_n,\frac{t}{|c|}\right) \text{ if } c \neq 0, c \in F,$$

(N5) for all $s,t \in \mathbb{R}$,

$$N(x_1,x_2,\cdots,x_n+x_n',s+t) \geq \min\{N(x_1,x_2,\cdots,x_n,s), N(x_1,x_2,\cdots,x_n',t)\},$$

(N6) $N(x_1,x_2,\cdots,x_n,\circ)$ is a non decreasing function of $\mathbb{R}$ and $\lim_{t\to\infty} N(x_1,x_2,\cdots,x_n,t) = 1$.
Then (X,N) is called a fuzzy n-normed linear space or in short f-n-NLS.

Remark 2.8 From (N3), it follows that in an f-n-NLS ,
(N4) for all $t \in \mathbb{R}$ with $t > 0$,

$$N(x_1,x_2,\cdots,cx_i,\cdots,x_n,t) = N\left(x_1,x_2,\cdots,x_i,\cdots,x_n,\frac{t}{|c|}\right) \text{ if } c \neq 0$$

(N5) for all $s,t \in \mathbb{R}$,

$$N(x_1,x_2,\cdots,x_i+x_i',\cdots,x_n,s+t) \geq \min\{N(x_1,x_2,\cdots,x_i,\cdots,x_n,s), N(x_1,x_2,\cdots,x_i',\cdots,x_n,t)\}.$$

Narayanan and Vijayabalaji have been proved an interesting notion of ascending family of α-n-norms corresponding to the fuzzy n-norm in the following theorem

Theorem 2.9 Let (X,N) be an f-n-NLS. Assume the condition that
(N7) $N(x_1,x_2,\cdots,x_n,t) > 0$ for all $t > 0$ implies $x_1,x_2,\cdots,x_n$ are linearly dependent.
Define $\|x_1,x_2,\cdots,x_n\|_\alpha = \inf\{t: N(x_1,x_2,\cdots,x_n,t) \geq \alpha\}, \alpha \in (0,1)$

Then $\{\|\cdot,\cdot,\cdots,\cdot\|_\alpha : \alpha \in (0,1)\}$ is an ascending family of n-norms on X. This n-norms are called α-n-norms on X corresponding to the fuzzy n-norms on X.

The following results is obtained from journal written by Gunawan and Mashadi

Definition 2.10 Let a sequence $\{x(k)\}$ in n-normed space $(X, \|\cdot,\cdots,\cdot\|)$ is said *converge* to an $x \in X$ (in the n-norm) whenever

$$\lim_{k\to\infty} \|x_1, \cdots, x_{n-1}, x(k) - x\| = 0$$

for every $x_1, \cdots, x_{n-1} \in X$. It is denoted by $x(k) \to x$ as $k \to \infty$.

Definition 2.11 Let a sequence $\{x(k)\}$ in n-normed space $(X, \|\cdot,\cdots,\cdot\|)$ is called *Cauchy sequence* (with respect to the n-norm) if

$$\lim_{k,l\to\infty} \|x_1, \cdots, x_{n-1}, x(k) - x(l)\| = 0$$

for every $x_1, \cdots, x_{n-1} \in X$.

If every Cauchy sequence in X converges to an $x \in X$, then X is said to be *complete* (with respect to the n-norm). A complete n-normed space is then called an n-Banach space.

3. Main Results

Inspired by a complete n-normed space, that is, n-Banach space, I will introduce the notion of fuzzy on n-Banach space.

Definition 3.1 A sequence $\{x(k)\}$ in a f-n-NLS (X, N) is called a Cauchy sequence with respect to α-n-norm if there exists $x_1, x_2, \cdots, x_n \in X$ and $y_1, y_2, \cdots, y_n \in X$ which are linearly independent such that $\lim_{k,l\to\infty} \|x_1, x_2, \cdots, x_n, x(k) - x(l)\|_\alpha = 0$ and $\lim_{k,l\to\infty} \|y_1, y_2, \cdots, y_n, x(k) - x(l)\|_\alpha = 0$

Theorem 3.2 Let (X, N) be a f-n-NLS and let $x_1, x_2, \cdots, x_n \in X$, $y_1, y_2, \cdots, y_n \in X$ and $z_1, z_2, \cdots, z_n \in X$.

(i) If $\{x(k)\}$ is a Cauchy sequence in (X, N) with respect to α-n-norm then $\{\|x_1, x_2, \cdots, x_n, x(k)\|_\alpha\}$ and $\{\|y_1, y_2, \cdots, y_n, x(k)\|_\alpha\}$ are real Cauchy sequences.

(ii) If $\{x(k)\}$ and $\{y(k)\}$ are Cauchy sequences in (X, N) with respect to α-n-norm and $\{\alpha(k)\}$ is a real Cauchy sequence then $\{x(k) + y(k)\}$ and $\{\alpha(k)x(k)\}$ are Cauchy sequences in (X, N) with respect to α-n-norm where $\alpha(k) \in [0,1]$.

Proof.

(i) $\|x_1, x_2, \cdots, x_n, x(k)\|_\alpha = \|x_1, x_2, \cdots, x_n, x(k) - x(l) + x(l)\|_\alpha$ therefore
$\|x_1, x_2, \cdots, x_n, x(k)\|_\alpha \le \|x_1, x_2, \cdots, x_n, x(k) - x(l)\|_\alpha + \|x_1, x_2, \cdots, x_n, x(l)\|_\alpha$
Also

$$\|x_1, x_2, \cdots, x_n, x(k)\|_\alpha - \|x_1, x_2, \cdots, x_n, x(l)\|_\alpha \le \|x_1, x_2, \cdots, x_n, x(k) - x(l)\|_\alpha$$

Therefore $\{\|x_1, x_2, \cdots, x_n, x(k)\|_\alpha\}$ is a real Cauchy sequence since the $\lim \|x_1, x_2, \cdots, x_n, x(k) - x(l)\|_\alpha = 0$

Similarly $\{\|y_1, y_2, \cdots, y_n, x(k)\|_\alpha\}$ is also a real Cauchy sequence.

(ii) $\left\|x_1, x_2, \cdots, x_n, \left(x(k) + y(k)\right) - \left(x(l) + y(l)\right)\right\|_\alpha = \left\|x_1, x_2, \cdots, x_n, \left(x(k) - x(l)\right) + \left(y(k) - y(l)\right)\right\|_\alpha$

$$\le \|x_1, x_2, \cdots, x_n, x(k) - x(l)\|_\alpha + \|x_1, x_2, \cdots, x_n, y(k) - y(l)\|_\alpha \to 0$$

Similarly $\left\|y_1, y_2, \cdots, y_n, \left(x(k) + y(k)\right) - \left(x(l) + y(l)\right)\right\|_\alpha \to 0$. Therefore $\{\|x_1, x_2, \cdots, x_n, x(k) + y(k)\|_\alpha\}$ is a Cauchy sequence on (X, N) with respect to α-n-norm. Also ,

$\|x_1, x_2, \cdots, x_n, \alpha(k)x(k) - \alpha(l)x(l)\|_\alpha$
$= \|x_1, x_2, \cdots, x_n, \alpha(k)x(k) - \alpha(k)x(l) + \alpha(k)x(l) - \alpha(l)x(l)\|_\alpha$

ISBN. 978-979-1222-95-2

$$= \left\| x_1, x_2, \cdots, x_n, \alpha(k)\big(x(k) - x(l)\big) + \big(\alpha(k) - \alpha(l)\big)x(l) \right\|_\alpha$$
$$\le \left\| x_1, x_2, \cdots, x_n, \alpha(k)\big(x(k) - x(l)\big) \right\|_\alpha + \left\| x_1, x_2, \cdots, x_n, \big(\alpha(k) - \alpha(l)\big)x(l) \right\|_\alpha$$
$$= |\alpha(k)| \|x_1, x_2, \cdots, x_n, x(k) - x(l)\|_\alpha + |\alpha(k) - \alpha(l)| \|x_1, x_2, \cdots, x_n, x(l)\|_\alpha$$
$$\to 0.$$

Since $\{\alpha(k)\}$ and $\{\|x_1, x_2, \cdots, x_n, x(k)\|_\alpha\}$ are real Cauchy sequences. Similarly $\|y_1, y_2, \cdots, y_n, \alpha(k)x(k) - \alpha(l)x(l)\|_\alpha \to 0$. Therefore $\{\alpha(k)x(k)\}$ is a Cauchy sequence in (X, N) with respect α-n-norm. ■

In [6], a detailed study about concepts for sequence in a fuzzy 2-normed space with respect to α-2-norm in X. In this part, we will introduce the notion of sequence in a fuzzy n-normed space with respect to α-n-norm.

Definition 3.3 A sequence $\{x(k)\}$ in a f-n-NLS (X, N) is said to converge to x if $\|x_1, x_2, \cdots, x_n, x(k) - x\|_\alpha \to 0$ as $k \to \infty$ with respect to α-n-norm for all $x_1, x_2, \cdots, x_n \in X$.

Theorem 3.4 In the f-n-NLS (X, N), satisfied

(i) If $x(k) \to x$ and $y(k) \to y$, then $x(k) + y(k) \to x + y$,
(ii) If $x(k) \to x$ and $\alpha(k) \to \alpha$, then $\alpha(k)x(k) \to \alpha x$
(iii) If $\dim(X, N) \ge 2$, $x(k) \to x$ and $x(k) \to y$ then $x = y$ convergence is with respect to α-n-norm.

Proof.

(i) $\left\| x_1, x_2, \cdots, x_n, \big(x(k) + y(k) - (x + y)\big) \right\|_\alpha = \|x_1, x_2, \cdots, x_n, (x(k) - x) + (y(k) - y)\|_\alpha$
$\le \|x_1, x_2, \cdots, x_n, x(k) - x\|_\alpha + \|x_1, x_2, \cdots, x_n, y(k) - y\|_\alpha \to 0$. Therefore $x(k) + y(k) \to x + y$.

(ii) If $x_1, x_2, \cdots, x_n \in X$,
$$\|x_1, x_2, \cdots, x_n, \alpha(k)x(k) - \alpha x\|_\alpha$$
$$= \|x_1, x_2, \cdots, x_n, \alpha(k)x(k) - \alpha(k)x + \alpha(k)x - \alpha x\|_\alpha$$
$$\le \|x_1, x_2, \cdots, x_n, \alpha(k)x(k) - \alpha(k)x\|_\alpha + \|x_1, x_2, \cdots, x_n, \alpha(k)x - \alpha x\|_\alpha$$
$$= |\alpha| \|x_1, x_2, \cdots, x_n, x(k) - x\|_\alpha + |\alpha(k) - \alpha| \|x_1, x_2, \cdots, x_n, x\|_\alpha \to 0.$$
Since $\|x_1, x_2, \cdots, x_n, x(k) - x\|_\alpha \to 0$ and $|\alpha(k) - \alpha| \to 0$, it follows that $\alpha(k)x(k) \to \alpha x$.

(iii) For any $x_1, x_2, \cdots, x_n \in X$,
$$\|x_1, x_2, \cdots, x_n, x - y\|_\alpha$$
$$= \|x_1, x_2, \cdots, x_n, x(k) - x(k) + x - y\|_\alpha$$
$$\le \|x_1, x_2, \cdots, x_n, x(k) - y\|_\alpha + \|x_1, x_2, \cdots, x_n, -(x(k) - x)\|_\alpha$$
$$= \|x_1, x_2, \cdots, x_n, x(k) - y\|_\alpha + \|x_1, x_2, \cdots, x_n, x(k) - x\|_\alpha \to 0$$
Since $x(k) \to x$ and $x(k) \to y$. Hence $x - y$, and $x_1, x_2, \cdots, x_n$ are linearly dependent for all $x_1, x_2, \cdots, x_n \in X$. Since $\dim(X, N) \ge 2$, the possibility is $x - y$ can be linearly dependent for all $x_1, x_2, \cdots, x_n \in X$ implies that $x - y = 0$ which implies that $x = y$. ■

Theorem 3.5 Let (X, N) be f-n-NLS. If $\lim \|x_1, x_2, \cdots, x_n, x(k) - x(l)\|_\alpha = 0$, then $\{\|x_1, x_2, \cdots, x_n, x(k) - y\|_\alpha\}$ is convergent sequence for each $y \in X$.

Proof.

$$\|x_1, x_2, \cdots, x_n, x(k) - y\|_\alpha = \|x_1, x_2, \cdots, x_n, x(k) - x(l) + x(l) - y\|_\alpha$$
$$\le \|x_1, x_2, \cdots, x_n, x(k) - x(l)\|_\alpha + \|x_1, x_2, \cdots, x_n, x(l) - y\|_\alpha$$

Therefore,
$$\|x_1, x_2, \cdots, x_n, x(k) - y\|_\alpha - \|x_1, x_2, \cdots, x_n, x(l) - y\|_\alpha \le \|x_1, x_2, \cdots, x_n, x(k) - x(l)\|_\alpha$$
Also,
$$\big| \|x_1, x_2, \cdots, x_n, x(k) - y\|_\alpha - \|x_1, x_2, \cdots, x_n, x(l) - y\|_\alpha \big| \le \|x_1, x_2, \cdots, x_n, x(k) - x(l)\|_\alpha$$
Hence $\{\|x_1, x_2, \cdots, x_n, x(k) - y\|_\alpha - \|x_1, x_2, \cdots, x_n, x(l) - y\|_\alpha\}$ is a convergent sequence since $\lim \|x_1, x_2, \cdots, x_n, x(k) - x(l)\|_\alpha = 0$. ■

ISBN. 978-979-1222-95-2

So, know let we see the definition of a fuzzy n-Banach space.

Definition 3.6 The fuzzy n-normed linear space (X, N) in which every Cauchy sequence converges is called a complete fuzzy n-normed linear space. The fuzzy n-normed linear space (X, N) is a fuzzy n-Banach space with respect to α-n-norm for it is a complete fuzzy n-normed linear with respect to α-n-norm.

REFERENCES

[1] Al Narayanan and S. Vijayabalaji, *Fuzzy n-Normed Space*, International Journal of Mathemataics and Mathematical Sciences 25(2005) 3963-3977

[2] B. Surender Reddy and Hemen Dutta, *On Equivalence of n-Norms in n-Normed Spaces*, The Pacific Journal of Science and Technology, 11(2010), 233-238

[3] H. Gunawan and Mashadi, *On n-Normed Spaces*, International Journal of Mathemataics and Mathematical Sciences (2001) 631-639

[4] Mehmet Acikgoz, *A Review on 2-Normed Structures*, Int.Journal of Math. Analysis 1(2007) 187-191

[5] R. Saadati and S.M. Vaezpour, *Some Results on Fuzzy Banach Spaces*, J. Appl. Math & Computing 17(2005) 475-484

[6] R.M. Somasundaram and Thangaraj Beaula, *Some Aspects of 2-Fuzzy 2-Normed Linear Spaces*, Bulletin of the Malaysian Mathematical Sciences Society 32(2009) 211-221

[7] S.K Elagan, E.M.E. Zayed, and T.A. Nofal, *Some Remarks on Series in Fuzzy n-Normed Spaces*, International Mathematical Forum 5, 3(2010) 117-124