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THE MAXIMAL IDEAL OF LOCALIZATION OF RING POLYNOMIAL OVER DEDEKIND DOMAIN

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ABSTRACT

Let R be a Dedekind domain with infinitely many primes and $\langle f \rangle \subset R[X]$ a principal prime ideal which is not maximal. Let $\mathfrak{m}$ be a maximal ideal of $R[X]$ and $\mathfrak{n}$ be a maximal ideal of $R[X]/\langle f \rangle$. Then localization of $R[X]/\langle f \rangle$ at $\mathfrak{n}$ is principal if and only if there exist t in $R[X]_{\mathfrak{m}}$ such that $\mathfrak{m}R[X]_{\mathfrak{m}} = \langle t, \bar{f} \rangle$.

Keywords: *Dedekind domain, localization, maximal ideal*

INTRODUCTION

Integral domain R with field of fraction $Q(R)$ ia a Dedekind domain if R are Noetherian, integrally closed in $Q(R)$ and every nonzero prime ideal is maximal ideal of R. Some examples of Dedekind domains are the ring of integers, the polynomial ring $F[X]$ in one variable over any field F, and any other principal ideal domain, but not all Dedekind domains are principal ideal domains.

Localization is systematic method of adding multiplicative inverses to a ring. The localization of R by S can be denoted by $S^{-1}R$ or R_S . If R is integral domain with field of fractions $Q(R)$, and $\mathfrak{p}$ is prime ideal of R , then the localization of R at $\mathfrak{p}$ is the subring

$$R_{\mathfrak{p}} = \{\frac{r}{s} \in Q(R) \ ; r \in R \text{ and } s\ inR\backslash\mathfrak{p}\}$$

of $Q(R)$. It is a local ring, with maximal ideal $\mathfrak{p}R_{\mathfrak{p}}$. In this work we proof the following : Let R be a Dedekind domain with infinitely many primes and $\langle f \rangle \subset R[X]$ a principal prime ideal which is not maximal. Let $\mathfrak{m}$ be a maximal ideal of $R[X]$ and $\mathfrak{n}$ be a maximal ideal of $R[X]/\langle f \rangle$. Then localization of $R[X]/\langle f \rangle$ at $\mathfrak{n}$ is principal if and only if there exist t in $R[X]_{\mathfrak{m}}$ such that $\mathfrak{m}R[X]_{\mathfrak{m}} = \langle t, \bar{f} \rangle$.

DEDEKIND DOMAIN AND LOCALIZATION

A **Dedekind domain** is an integral domain satisfying the following three conditions :

1. R is Noetherian ring.
2. R is integrally closed.
3. Every nonzero prime ideal of R is maximal.

A Principal Ideal Domain (PID) satisfies all three conditions and therefore a Dedekind Domain. Hillman (1986) has proved that no maximal ideal of ring polynomial over a Dedekind domain is principal.

Let R be a ring and $S \subseteq R$ a **multiplicative set**; that is, suppose that :

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1. $S \neq \emptyset$ dan $1 \in S$
2. If $a, b \in S$ then $ab \in S$;.

Suppose that $f : R \to B$ is a ring homomorphism satisfying the two conditions :

1. $f(x)$ is a unit in B for all $x \in S$.
2. If $g : R \to B'$ is a homomorphism of rings taking every element of S to a unit of B' then there exist a unique homomorphism $h : B \to B'$ such that $g = hf$.

Ring B satisfying the previous conditions is called the **localization** or the **ring of fractions** of R with respect to S.We write $B = S^{-1}R$ or R_s, where

$$S^{-1}R = \{\frac{r}{s} \in Q(R)\ ; r \in R \text{ dan } s \in S\}$$

Now define the following operations on B

$$\begin{aligned} \forall \tfrac{a}{b}, \tfrac{a'}{b'} \in S^{-1}R \quad &\Longrightarrow \quad \tfrac{a}{b} + \tfrac{a'}{b'} = \tfrac{ab'+a'b}{bb'} \\ &\Longrightarrow \quad \tfrac{a}{b} \cdot \tfrac{a'}{b'} = \tfrac{aa'}{bb'} \end{aligned}$$

Furthermore, define $f : R \longrightarrow S^{-1}R$ with $f(a) = \frac{a}{1}$, for all $a \in R$, and f is a ring homomorphism.

Theorem 2.1. *(i) All the ideals of R_s are of the form IR_s, with I an ideal of R.*
(ii) Every prime ideal of R_s is of the form $\mathfrak{p}R_s$ with $\mathfrak{p}$ a prime ideal of R disjoint from S, and conversely, $\mathfrak{p}R_s$ is prime in R_s for every such $\mathfrak{p}$.

Let $\mathfrak{p}$ be a prime ideal of R, and set $S = R - \mathfrak{p}$. In this case we usually write $R_{\mathfrak{p}}$ for R_s. the localization $R_{\mathfrak{p}}$ is a local ring with maximal ideal $\mathfrak{p}R_{\mathfrak{p}}$. Indeed, as we saw in theorem before, $\mathfrak{p}R_{\mathfrak{p}}$ is a prime ideal of $R_{\mathfrak{p}}$, and furthermore, if $J \subset R_{\mathfrak{p}}$ is any proper ideal then $I = J \cap R$ is an ideal of R disjoint from $R - \mathfrak{p}$, and so $I \subset \mathfrak{p}$, giving $J = IR_{\mathfrak{p}} \subset \mathfrak{p}R_{\mathfrak{p}}$. The prime ideals of $R_{\mathfrak{p}}$ correspond bijectively with the prime ideals of R contained in $\mathfrak{p}$.

Theorem 2.2. *Let $\mathfrak{p}$ is a prime prim of R, then : (i) There is a one-one correspondence between the set of prime ideals of R contained in $\mathfrak{p}$ and the set of prime ideals of $R_{\mathfrak{p}}$.*
(ii) The ideal $\mathfrak{p}R_{\mathfrak{p}}$ in $R_{\mathfrak{p}}$ is the unique maximal ideal of $R_{\mathfrak{p}}$.

The result give us the following definition "A **local ring** is a commutative ring with identity which has a unique maximal ideal."

Example : 1. Every field is a local ring with $\{0\}$ is its maximal ideal. 2. if p is prime and $n \geq 1$, then $\mathbb{Z}_{p^n}$ is local ring with unique maximal ideal $\langle p \rangle$. Let R is Dedekind domain and $\langle f \rangle \subset R[X]$ a principal prime ideal which is not maximal. Let $\mathfrak{m}$ is maximal ideal of $R[X]$ and $R[X]_{\mathfrak{m}}$ is localization of $R[X]$ at $\mathfrak{m}$. Then, no maximal ideal of $R[X]_{\mathfrak{m}}$ is principal and $\mathfrak{m}R[X]_{\mathfrak{m}}/(\mathfrak{m}R[X]_{\mathfrak{m}})^2$ is two dimensional vektor space over the field $R[X]_{\mathfrak{m}}/\mathfrak{m}R[X]_{\mathfrak{m}}$.(Helmi, 2009).

THE MAXIMAL IDEAL OF LOCALIZATION OF RING POLYNOMIAL OVER DEDEKIND DOMAIN

We will start with a lemma before given proof of our main theorems. These lemmas will be used to proof our first main theorem.

Now, let R be a Dedekind domain with infinitely many prime ideals. And let $\mathfrak{m}$ and $\mathfrak{n}$ be a maximal ideal of $R[X]$ and S, respectively. Let $R[X]_{\mathfrak{m}}$ be a localization of R at $\mathfrak{m}$ and $S_{\mathfrak{n}}$ be a localization of of S at $\mathfrak{n}$. Then the following diagram will be used to proof our theorem.

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$$\begin{array}{rcccc} \pi_1 & : & R[X] & \longrightarrow & R[X]_{\mathfrak{m}} \\ & & r(x) & \longmapsto & \bar{r}(x) = \frac{r(x)}{1} \\ & & \downarrow & & \\ \pi_2 & : & S = R[X]/\langle f \rangle & \longrightarrow & S_{\mathfrak{n}} = [R[X]/\langle f \rangle]_{\mathfrak{n}} \end{array}$$

Lemma 3.1. *Maximal ideal of $R[X]$ which is contained f, is one-one correspondence with maximal ideal of $S = R[X]/\langle f \rangle$.*

Proof. See Helmi[3] for the details. ∎

Now, we state our main theorems. **Theorem 2.1.** *Let R be a Dedekind domain and $\langle f \rangle \subset R[X]$ be a principal ideal which is not maximal. Let $\mathfrak{m}$ be a maximal ideal of $R[X]$. If $f \in \mathfrak{m}$ maps to $\bar{f} \in R[X]_{\mathfrak{m}}$, then there exist a ring homomorphism $R[X]_{\mathfrak{m}}/\langle \bar{f} \rangle \cong S_{\mathfrak{n}} = [R[X]/\langle f \rangle]_{\mathfrak{n}}$, for all $\mathfrak{n}$, the maximal ideal of $R[X]/\langle f \rangle$.*

Proof. Let us define a map,

$$\begin{array}{rcccc} \theta & : & R[X]_{\mathfrak{m}}/\langle \bar{f} \rangle & \longrightarrow & [R[X]/\langle f \rangle]_{\mathfrak{n}} \\ & & \frac{r(x)}{s(x)} + \langle \bar{f} \rangle & \longmapsto & \frac{r(x)+\langle f \rangle}{s(x)+\langle f \rangle} \end{array}$$

where $s(x) \notin \mathfrak{m}$ and $s(x) + \langle f \rangle \notin \mathfrak{n}$. First, we will proved that $\frac{r(x)+\langle f \rangle}{s(x)+\langle f \rangle} = \frac{r(x)}{s(x)} + \langle f \rangle$ .Since

$$\frac{r(x) + \langle f \rangle}{s(x) + \langle f \rangle} = (r(x) + \langle f \rangle)(s(x) + \langle f \rangle)^{-1}$$

and

$$\left(\frac{1}{s(x)} + \langle f \rangle\right)(s(x) + \langle f \rangle) = 1 + \langle f \rangle$$

then $(s(x) + \langle f \rangle)^{-1} = \left(\frac{1}{s(x)} + \langle f \rangle\right)$, therefore

$$\begin{aligned} \frac{r(x) + \langle f \rangle}{s(x) + \langle f \rangle} &= (r(x) + \langle f \rangle)(s(x) + \langle f \rangle)^{-1} \\ &= (r(x) + \langle f \rangle)\left(\frac{1}{s(x)} + \langle f \rangle\right) \\ &= \frac{r(x)}{s(x)} + \langle f \rangle \end{aligned}$$

Suppose $f = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n$ maps to $\bar{f} = \frac{f}{1}$. Let $\frac{r_1(x)}{s_1(x)} + \langle \bar{f} \rangle, \frac{r_2(x)}{s_2(x)} + \langle \bar{f} \rangle \in R[X]_{\mathfrak{m}}/\langle \bar{f} \rangle$, and $\frac{r_1(x)}{s_1(x)} + \langle \bar{f} \rangle = \frac{r_2(x)}{s_2(x)} + \langle \bar{f} \rangle$, then

$$\begin{aligned} \frac{r_1(x)}{s_1(x)} + \langle \bar{f} \rangle &= \frac{r_2(x)}{s_2(x)} + \langle \bar{f} \rangle \\ \frac{r_1(x)}{s_1(x)} - \frac{r_2(x)}{s_2(x)} &\in \langle \bar{f} \rangle \\ \frac{r_1(x)}{s_1(x)} - \frac{r_2(x)}{s_2(x)} &= h(x).\bar{f} \\ &= h(x).\frac{f}{1} \\ &= h(x).f \\ &= \langle f \rangle \\ \frac{r_1(x)}{s_1(x)} + \langle f \rangle &= \frac{r_2(x)}{s_2(x)} + \langle f \rangle \end{aligned}$$

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Hence, θ is well defined. For any $a = \frac{r_1(x)}{s_1(x)} + \langle \bar{f} \rangle, b = \frac{r_2(x)}{s_2(x)} + \langle \bar{f} \rangle \in R[X]_m / \langle \bar{f} \rangle$ then

$$\begin{aligned}
\theta(a+b) &= \theta\left(\left(\frac{r_1(x)}{s_1(x)} + \langle \bar{f} \rangle\right) + \left(\frac{r_2(x)}{s_2(x)} + \langle \bar{f} \rangle\right)\right) \\
&= \theta\left(\left(\frac{r_1(x)}{s_1(x)} + \frac{r_2(x)}{s_2(x)}\right) + \langle \bar{f} \rangle\right) \\
&= \theta\left(\frac{r_1(x).s_2(x) + r_2(x).s_1(x)}{s_1(x).s_2(x)} + \langle \bar{f} \rangle\right) \\
&= \frac{r_1(x).s_2(x) + r_2(x).s_1(x) + \langle f \rangle}{s_1(x).s_2(x) + \langle f \rangle} \\
&= \left(\frac{r_1(x).s_2(x) + r_2(x).s_1(x)}{s_1(x).s_2(x)}\right) + \langle f \rangle \\
&= \left(\frac{r_1(x)}{s_1(x)} + \frac{r_2(x)}{s_2(x)}\right) + \langle f \rangle \\
&= \theta(a) + \theta(b)
\end{aligned}$$

and

$$\begin{aligned}
\theta(ab) &= \theta\left(\left(\frac{r_1(x)}{s_1(x)} + \langle \bar{f} \rangle\right) \cdot \left(\frac{r_2(x)}{s_2(x)} + \langle \bar{f} \rangle\right)\right) \\
&= \theta\left(\left(\frac{r_1(x)}{s_1(x)} \cdot \frac{r_2(x)}{s_2(x)}\right) + \langle \bar{f} \rangle\right) \\
&= \theta\left(\left(\frac{r_1(x).r_2(x)}{s_1(x).s_2(x)}\right) + \langle \bar{f} \rangle\right) \\
&= \frac{r_1(x).r_2(x) + \langle f \rangle}{s_1(x).s_2(x) + \langle f \rangle} \\
&= \frac{r_1(x).r_2(x)}{s_1(x).s_2(x)} + \langle f \rangle \\
&= \left(\frac{r_1(x)}{s_1(x)} \cdot \frac{r_2(x)}{s_2(x)}\right) + \langle f \rangle \\
&= \left(\frac{r_1(x)}{s_1(x)} + \langle f \rangle\right) \cdot \left(\frac{r_2(x)}{s_2(x)} + \langle f \rangle\right) \\
&= \theta(a).\theta(b)
\end{aligned}$$

and also

$$\theta\left(\frac{1}{1} + \langle \bar{f} \rangle\right) = \frac{1 + \langle f \rangle}{1 + \langle f \rangle}$$

Hence, θ is a ring homomorphism. Suppose $\theta(a) = \frac{r_1(x)+\langle f \rangle}{s_1(x)+\langle f \rangle}$ and $\theta(b) = \frac{r_2(x)+\langle f \rangle}{s_2(x)+\langle f \rangle}$ where $\theta(a) = \theta(b)$. Since

$$\frac{r_1(x) + \langle f \rangle}{s_1(x) + \langle f \rangle} = \frac{r_1(x)}{s_1(x)} + \langle f \rangle \text{ and } \frac{r_2(x) + \langle f \rangle}{s_2(x) + \langle f \rangle} = \frac{r_2(x)}{s_2(x)} + \langle f \rangle$$

thus,

$$\frac{r_1(x) + \langle f \rangle}{s_1(x) + \langle f \rangle} = \frac{r_2(x) + \langle f \rangle}{s_2(x) + \langle f \rangle}$$

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hence

$$\begin{aligned}
\frac{r_1(x)}{s_1(x)} + \langle f\rangle &= \frac{r_2(x)}{s_2(x)} + \langle f\rangle \\
\frac{r_1(x)}{s_1(x)} - \frac{r_2(x)}{s_2(x)} &\in \langle f\rangle \\
\frac{r_1(x)}{s_1(x)} - \frac{r_2(x)}{s_2(x)} &= h(x).f \\
&= h(x)\left(a_0 + a_1x + a_2x^2 + \ldots + a_nx^n\right) \\
&= h(x)\frac{\left(a_0 + a_1x + a_2x^2 + \ldots + a_nx^n\right)}{1} \\
&= h(x).\frac{f}{1} \\
&= \langle \bar{f}\rangle \\
\frac{r_1(x)}{s_1(x)} + \langle \bar{f}\rangle &= \frac{r_2(x)}{s_2(x)} + \langle \bar{f}\rangle
\end{aligned}$$

So, we have shown that θ is injective. Finally, for any $\frac{r(x)+\langle f\rangle}{s(x)+\langle f\rangle} \in [R[X]/\langle f\rangle]_{\mathfrak{n}}$, and since maxsimal ideal $\mathfrak{n}$ is one-one correspondence with $\mathfrak{m}$, hence, $s(x) \notin \mathfrak{m}$. Now,

$$\frac{r(x)+\langle f\rangle}{s(x)+\langle f\rangle} = \frac{r(x)}{s(x)} + \langle f\rangle$$

then, for all $\frac{r(x)+\langle f\rangle}{s(x)+\langle f\rangle} \in [R[X]/\langle f\rangle]_{\mathfrak{n}}$ there exist $\frac{r(x)}{s(x)} + \langle \bar{f}\rangle \in R[X]_{\mathfrak{m}}/\langle \bar{f}\rangle$, such that $\theta\left(\frac{r(x)}{s(x)} + \langle \bar{f}\rangle\right) = \frac{r(x)+\langle f\rangle}{s(x)+\langle f\rangle}$, which is showed that $R[X]_{\mathfrak{m}}/\langle \bar{f}\rangle \cong S_{\mathfrak{n}} = [R[X]/\langle f\rangle]_{\mathfrak{n}}$. ■

Theorem 3.3. *Let R is Dedekind domain and $\langle f\rangle \subset R[X]$ a principal prime ideal which is not maximal. Let $\mathfrak{m}$ is maximal ideal of $R[X]$ and $\mathfrak{n}$ is maximal ideal $S = R[X]/\langle f\rangle$. Then localization of S at $\mathfrak{n}$ is principal if and only if there exist t in $R[X]_{\mathfrak{m}}$ suth that $\mathfrak{m}R[X]_{\mathfrak{m}} = \langle t, \bar{f}\rangle$.*

Proof. Since $S_{\mathfrak{n}} \cong R[X]_{\mathfrak{m}}/\langle \bar{f}\rangle$, then maximal ideal of $S_{\mathfrak{n}}$, which is $\mathfrak{n}S_{\mathfrak{n}}$ isomorphic to $\mathfrak{m}R[X]_{\mathfrak{m}}/\langle \bar{f}\rangle$. Suppose that $\mathfrak{m}R[X]_{\mathfrak{m}}/\langle \bar{f}\rangle$ is principal ideal. Let $\mathfrak{m}R[X]_{\mathfrak{m}}/\langle \bar{f}\rangle = \langle t + \langle \bar{f}\rangle\rangle$, then $\mathfrak{m}R[X]_{\mathfrak{m}}/\langle \bar{f}\rangle = \left\{(a + \langle \bar{f}\rangle)(t + \langle \bar{f}\rangle);\ a \in R[X]_{\mathfrak{m}}\right\}$. Let $y \in \mathfrak{m}R[X]_{\mathfrak{m}}$, hence, $y + \langle \bar{f}\rangle \in \mathfrak{m}R[X]_{\mathfrak{m}}/\langle \bar{f}\rangle$, thus,

$$\begin{aligned}
y + \langle \bar{f}\rangle &= \left((a + \langle \bar{f}\rangle)(t + \langle \bar{f}\rangle)\right) \\
&= at + \langle \bar{f}\rangle
\end{aligned}$$

Hence,

$$y - at \in \langle \bar{f}\rangle$$

and

$$\begin{aligned}
y - at &= b\bar{f} \\
y &= at + b\bar{f}
\end{aligned}$$

where $a, b \in R[X]_{\mathfrak{m}}$, and $t, \bar{f} \in \mathfrak{m}R[X]_{\mathfrak{m}}$. Since $y \in \mathfrak{m}R[X]_{\mathfrak{m}}$ hence $\mathfrak{m}R[X]_{\mathfrak{m}} = \langle t, \bar{f}\rangle$. Conversely, suppose there exists $t \in R[X]_{\mathfrak{m}}$ such that $\mathfrak{m}R[X]_{\mathfrak{m}} = \langle t, \bar{f}\rangle$. For any $w \in R[X]_{\mathfrak{m}}$, we have

$$\begin{aligned}
w &= \langle t, \bar{f}\rangle \\
w &= at + b\bar{f};\quad a, b \in R[X]_{\mathfrak{m}} \\
w - at &= b\bar{f} \\
w - at &\in \langle \bar{f}\rangle
\end{aligned}$$

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Hence

$$\begin{aligned} w + \langle \bar{f} \rangle &= at + \langle \bar{f} \rangle \\ &= (a + \langle \bar{f} \rangle)(t + \langle \bar{f} \rangle) \end{aligned}$$

Thus

$$w + \langle \bar{f} \rangle \in \langle t + \langle \bar{f} \rangle \rangle$$

It is clear that $w + \langle \bar{f} \rangle \in \mathfrak{m}R[X]_\mathfrak{m} / \langle \bar{f} \rangle$. Hence, $\mathfrak{m}R[X]_\mathfrak{m} / \langle \bar{f} \rangle$ is principal. ∎

CONCLUDING REMARKS

The main theorem in this work can be used to proof the main theorem in [2], which is, if R is a Dedekind domain and f generates a prime ideal of $R[X]$ which is not maximal, then the domain $R[X]/\langle f \rangle$ is Dedekind if and only f is not contained in the square of any maximal ideal of $R[X]$

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REFERENCES

1. D.A. Passman, *A Course in Ring Theory*, Wadsworth Inc, California, 1991.
2. J.A. Hillman, Polynomial Determining Dedekind Domain, *Bull. Austral. Math. Soc*, **29**, 167–175, 1984.
3. M.R. Helmi, *Polynomial Over Dedekind Domain*, Proceeding of IICMA, 2009.
4. M.R. Helmi, *Ruang Vektor Berdimensi Dua Dari Lokalisasi Gelanggang Suku Banyak*, Prosiding Semirata BKS-PTN Wilayah Barat Ke-23, 2010.
5. T.W. Hungerford, *Algebra*, Springer-Verlag, New-York, 1974.
6. T.Y. Lam, *Graduate Text in Mathematics : A First Course in Noncommutative Rings*, Springer-Verlag, New-York, 1991.