TWO STEP METHOD WITHOUT EMPLOYING DERIVATIVES FOR SOLVING A NONLINEAR EQUATION

M. Imran, Agusni, A. Karma, S. Putra

Abstract. *We discuss an iterative method for finding root of a nonlinear equation employing central differences to avoid derivatives in the method. We show that this two step method is of order three. Numerical simulations show that the method is comparable with others third order methods.*

1. INTRODUCTION

Finding the root of a nonlinear equation, $f(x) = 0$, is a classic problem in numerical analysis. Recently, many iterative method have been developed to solve a nonlinear equation by combining two or more existing method [2, 6, 8, 10, 11]. Kasturiarachi [5] combines Newton's method and secant method, that is

$$x_n^* = x_n - \frac{f(x_n)}{f'(x_n)} \tag{1}$$

$$x_{n+1} = x_n - \frac{f^2(x_n)}{f'(x_n)(f(x_n) - f(x_n^*))}. \tag{2}$$

It is shown that this is a third order method. Combination of Secant method and Newton's method has been discussed without analyzing the order of convergence of the method by Demidovich and Maron (See[3], Sec. 4.7).

To avoid the derivative appearing in (1) and (2), Jain [4] approximates

$$f'(x_n) = \frac{f(x_n + f(x_n)) - f(x_n)}{f(x_n)}$$

so that equation(1) becomes

$$x_n^* = x_n - \frac{f^2(x_n)}{f(x_n + f(x_n)) - f(x_n)} \tag{3}$$

which is Steffensen's method, a good competitive for Newton's method without employing a derivative in its formula. Using this strategy, he proposes the the third order method using three function evaluation per iteration as follows:

$$x_n^* = x_n - \frac{f^2(x_n)}{f(x_n + f(x_n)) - f(x_n)} \tag{4}$$

$$x_{n+1} = x_n - \frac{f^3(x_n)}{(f(x_n + f(x_n)) - f(x_n))(f(x_n) - f(x_n^*))}. \tag{5}$$

The aim of this paper is to propose an alternative method of order three without employing derivative in its formulae and to do some numerical comparisons with some available third order methods.

2. PRELIMINARY RESULTS

Definition 2.1 (See [1]) *A sequence of iterates $\{x_n : n \geq 0\}$ is said to converge with order $p \geq 1$ to a point α if*

$$|\alpha - x_{n+1}| \leq c|\alpha - x_n|^p, \quad n \geq 0$$

for some $c > 0$. If $p = 1$, the sequence is said to converge linearly to α. In that case, we require $c < 1$; the constant c is called the rate of linear convergence of x_n to α.

Definition 2.2 (See [11]) *Let α be a root of the function f and suppose that x_{n-1}, x_n, x_{n+1} are closer to the root α. Then the computational order of convergence p can be approximated using the formula*

$$p \approx \frac{\ln |(x_{n+1} - \alpha)/(x_n - \alpha)|}{\ln |(x_n - \alpha)/(x_{n-1} - \alpha)|} \tag{6}$$

3. PROPOSED METHOD

If we approximate the derivative appearing in (1) and (2) with central difference, that is

$$f'(x_n) = \frac{f(x_n + f(x_n)) - f(x_n - f(x_n))}{2f(x_n)}$$

we have the following iterative formulae

$$x_n^* = x_n - \frac{2f^2(x_n)}{f(x_n + f(x_n)) - f(x_n - f(x_n))} \tag{7}$$

$$x_{n+1} = x_n - \frac{2f^3(x_n)}{(f(x_n + f(x_n)) - f(x_n - f(x_n)))(f(x_n) - f(x_n^*))}. \tag{8}$$

We prove below that the iterative method (7) and (8) is of order 3.

Theorem 3.1 *Let $f : D \subset \mathbb{R} \to \mathbb{R}$ for an open interval D. Assume that f has first, second, and third derivatives in the interval D. If f has a simple root at $\alpha \in D$ and x_0 is sufficiently close to α, then the new method defined by (7) and (8) satisfies the following error equation:*

$$e_{n+1} = c_2^2 e_n^3 + \mathcal{O}(e_n^4),$$

where $e_n = x_n - \alpha$ and

$$c_j = \frac{f^{(j)}(\alpha)}{j!\, f'(\alpha)}, \quad j = 2, 3.$$

Proof:
Let α be a simple root of $f(x) = 0$, and $x_n = \alpha + e_n$. Denote

$$F_i = f^{(i)}(\alpha), \ i = 1, 2, 3 \quad \text{so that} \quad c_j = \frac{F_j}{j! F_1}, \ i = 2, 3.$$

Taylor expansion of $f(x_n)$, $f(x_n + f(x_n))$ and $f(x_n - f(x_n))$ about $x = \alpha$, which is a zero of f, is

$$f(x_n) = F_1(e_n + c_2 e_n^2 + c_3 e_n^3 + \mathcal{O}(e_n^4)), \tag{9}$$

$$f(x_n + f(x_n)) = F_1 \Big[(1 + F_1)e_n + \Big(\tfrac{3F_2}{2} + c_2 + \tfrac{F_1 F_2}{2} \Big) e_n^2$$
$$+ \Big(\tfrac{2F_3}{3} + c_2 F_2 + c_3 + \tfrac{F_1 F_3}{2} + \tfrac{F_3 F_1}{6} \Big) e_n^3 + \mathcal{O}(e_n^4) \Big], \tag{10}$$

and

$$f(x_n - f(x_n)) = F_1\Big[(-1 + F_1)e_n + \Big(\tfrac{-3F_2}{2} + c_2 + \tfrac{F_1 F_2}{2}\Big)e_n^2$$
$$+ \Big(\tfrac{-2F_3}{3} - c_2 F_2 + c_3 + \tfrac{F_1 F_3}{2} + \tfrac{F_3 F_1}{6}\Big)e_n^3 + \mathcal{O}(e_n^4)\Big]. \quad (11)$$

Using (9)–(11), we obtain after simplifying

$$\frac{2f^2(x_n)}{f(x_n + f(x_n)) - f(x_n)} = \frac{(2c_3 + c_2^2)e_n^3 + 2c_2 e_n^2 + e_n + \mathcal{O}(e_n^4)}{1 + 3c_2 e_n + (2c_2^2 + \tfrac{c_2}{2F_1} + \tfrac{F_1 F_3}{6} + 4c_3 - \tfrac{c_3}{2F_1}) + \mathcal{O}(e_n^4)} \quad (12)$$

Recalling

$$(1 + x)(1 - x + x^2 - x^3 + x^4 - x^5 + \cdots + \mathcal{O}(x^n)) = 1 + \mathcal{O}(x^n) \quad (13)$$

and after some algebra we can express (12) as

$$\frac{2f^2(x_n)}{f(x_n + f(x_n)) - f(x_n)} = e_n - c_2 e_n^2 + \Big(-2c_3 - \tfrac{F_1 F_3}{6} + 2c_2^2 - \tfrac{c_2}{2F_1}\Big)e_n^3 + \mathcal{O}(e_n^4). \quad (14)$$

On substituting (14) into (7) and simplifying yields

$$x_n^* = \alpha + c_2 e_n^2 - \Big(-2c_3 - \frac{F_1 F_3}{6} + 2c_2^2 - \frac{c_2}{2F_1}\Big)e_n^3 + \mathcal{O}(e_n^4). \quad (15)$$

Expanding $f(x_n^*)$ about $x = \alpha$, computing $\frac{f(x_n)}{f(x_n) - f(x_n^*)}$, noting (13) and simplifying, we end up with

$$\frac{f(x_n)}{f(x_n) - f(x_n^*)} = 1 + c_2 e_n + \Big(2c_3 - 2c_2^2 + \tfrac{c_2}{2F_1} + \tfrac{F_1 F_3}{6} - \tfrac{c_3}{2F_1}\Big)e_n^2$$
$$+ \Big(-2c_2^3 + \tfrac{c_2 F_1 F_3}{6} - \tfrac{c_2 c_3}{2F_1} + c_2 c_3 + \tfrac{c_2^2}{2F_1}\Big)e_n^3 + \mathcal{O}(e_n^4). \quad (16)$$

Furthermore, adding (14) and (16), and simplifying, we obtain

$$\frac{2f^3(x_n)}{(f(x_n + f(x_n)) - f(x_n - f(x_n)))(f(x_n) - f(x_n^*))} = e_n - c_2^2 e_n^3 + \mathcal{O}(e_n^4). \quad (17)$$

On substituting (17) into (8), we obtain, the following equation error

$$e_{n+1} = c_2^2 e_n^3 + \mathcal{O}(e_n^4).$$

This ends the proof.

4. NUMERICAL COMPARISONS

In this section we present numerical comparisons of proposed method (SS), Newton method (NM), Newton Secant method (NS) and Jain method (JM) to solve a nonlinear equation. We see the number of iteration of each method for a given accuracy and the COC of each method. We stop the iteration process if one of the following criteria satisfied

(a) $f'(x_n) = 0$, for NM and NS methods.

(b) Maximum iteration=1000.

(c) If $|f(x_{n+1})| < 10^{-10}$.

(d) If $|x_{n+1} - x_n| < x_{n+1} \times 10^{-10}$.

We use the following test functions:

$$f_1(x) = x^3 - 3x^2 - 5, \qquad \alpha = 3.42598875736162212607, [5]$$
$$f_2(x) = x^4 - 3x^3 - 54x^2 - 150x - 100, \qquad \alpha = -1.0, [7]$$
$$f_3(x) = 0.5 + \sin(x), \qquad \alpha = -0.52359877559829887307, [7]$$
$$f_4(x) = x \exp(2x), \qquad \alpha = 0.0.$$

In the Table 1, NA stands for the method is not applicable, NC is the method is not convergent to the given root, and the start * in the number of iteration in the third nonlinear function indicates that the method is convergent to a different root. From Table 1, we see that for the first and fourth functions, in terms of the number of iteration, SS-method method is better than others mentioned methods. In most cases, The SS-method is better than JM-method, in term of the number of iteration. All the mention methods cannot differentiate between the roots and the asymptotic trend of the function as occurs for initial guesses $-4.0, -2.0, -1.0$ in the third nonlinear function. Overall SS-method is comparable with JM-method.

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Table 1: Comparisons of the number of iterations and the COC of the discused methods

$f(x)$	x_0	Number of Iterations				COC			
		NM	NS	JM	SS	NM	NS	JM	SS
$x^3 - 3x^2 - 5$	-2.0	15	10	34	16	2.00	2.86	3.06	3.15
	-0.9	11	>1000	22	13	2.00	$-$	3.01	3.20
	0.0	NA	NA	18	16	$-$	$-$	2.95	3.23
	1.5	10	5	16	12	2.00	3.01	2.99	3.35
	5.0	6	4	5	5	2.00	2.99	2.86	3.24
$x^4 - 3x^3 -$ $54x^2 -$ $150x - 100$	-1.9	9	6	9	8	2.00	3.12	3.20	3.35
	-1.6	7	4	6	6	2.08	3.12	3.13	3.29
	-1.2	5	4	4	4	2.00	3.13	3.30	3.49
	-0.4	5	4	5	5	2.00	2.99	3.09	3.24
	0.0	6	4	5	5	2.00	2.96	2.65	3.54
$0.5 + \sin(x)$	-1.0	5	3	4	3	2.00	2.48	3.03	3,41
	-0.5	3	2	2	2	1.99	INF	2.5	INF
	0.0	4	3	3	3	1.99	INF	2.4	INF
	1.0	7*	4	4*	5*	2.00	3.17	3.51	3.01
	1.5	6*	6*	3*	7*	2.00	2.49	2.62	INF
$x \exp(2x)$	-4.0	NC	NC	NC	NC	$-$	$-$	$-$	$-$
	-2.0	NC	NC	NC	NC	$-$	$-$	$-$	$-$
	-1.0	NC	NC	NC	NC	$-$	$-$	$-$	$-$
	-0.3	8	5	7	5	2.00	2.90	3.00	3.02
	0.3	6	4	4	4	2.00	2.97	3.00	3.01

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M. Imran: Laboratorium Matematika Terapan, Jurusan Matematika, Fakultas Matematika dan Ilmu Pengetahuan Alam, Universitas Riau, Kampus Binawidya Pekanbaru (28293), Indonesia.
E-mail: mimran@unri.ac.id

Agusni: Laboratorium Matematika Terapan, Jurusan Matematika, Fakultas Matematika dan Ilmu Pengetahuan Alam, Universitas Riau, Kampus Binawidya Pekanbaru (28293), Indonesia.
E-mail: agusni@yahoo.co.id

A. Karma: Laboratorium Matematika Terapan, Jurusan Matematika, Fakultas Matematika dan Ilmu Pengetahuan Alam, Universitas Riau, Kampus Binawidya Pekanbaru (28293), Indonesia.
E-mail: smintan@yahoo.com

S. Putra: Laboratorium Matematika Terapan, Jurusan Matematika, Fakultas Matematika dan Ilmu Pengetahuan Alam, Universitas Riau, Kampus Binawidya Pekanbaru (28293), Indonesia.
E-mail: sputra@unri.ac.id