Midpoint Newton's Method for Simple and Multiple Roots *

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Abstract

In this paper we propose a new modification of Newton's method based on midpoint rule for solving nonlinear equations. Analysis of convergence shows that the new method is cubically convergent for a simple root and linearly for multiple roots. The method require one function evaluation and two of its first derivative, but no evaluations of its second derivative. We verify the theoretical results on relevant numerical problems and compare the behavior of the propose method with some existing ones.

Keywords: *Modified Newton's Method, Midpoint rule, trapezoidal rule*

1 Introduction

Newton's method, which is quadratically convergent, is the most popular method to find a root of a nonlinear equation,

$$f(x) = 0, \qquad f : D \subset \mathbb{R} \to \mathbb{R}. \tag{1}$$

Many researcher are interested in modifying the method to obtain a higher order method. The first third order method resulted on the modifying the Newton's method appear in Wall [8]. This modification requires the second derivative of $f(x)$.

Weerakoon and Fernando [7] have suggested an improvement to the iteration of Newton's method without requiring the second derivative of $f(x)$. They have approximated the indefinite integral using trapezoidal rule.

In this study we suggest a modification of the iteration of Newton's method by approximating the indefinite integral using a midpoint rule. The modified method need one functional and two first derivative evaluations for each iteration.

2 Preliminary Results

Definition 1 (See [1]) A sequence of iterates $\{x_n : n \geq 0\}$ is said to converge with order $p \geq 1$ to a point α if

$$|\alpha - x_{n+1}| \leq c|\alpha - x_n|^p, \quad n \geq 0$$

for some $c > 0$. If $p = 1$, the sequence is said to converge linearly to α. In that case, we require $c < 1$; the constant c is called the rate of linear convergence of x_n to α.

Let $e_n = x_n - \alpha$ be the error in the nth iterate of the method which produce the sequence $\{x_n\}$. Then the relation

$$e_{n+1} = ce_n^p + \mathcal{O}(e_n^{p+1}) = \mathcal{O}(e_n^p)$$

is called the error equation. The value of p is called the order of convergence and c is known as the asymptotic error constant of this method.

*Presented at the 14th National Conference in Mathematics, and the Congress of Indonesian Mathematical Society, held at the Sriwijaya University in Palembang, July 24-27, 2008

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Definition 2 (See [7]) Let α be a root of the function f and suppose that x_{n-1}, x_n, x_{n+1} are closer to the root α. Then the computational order of convergence p can be approximated using the formula

$$p \approx \frac{\ln |(x_{n+1} - \alpha)/(x_n - \alpha)|}{\ln |(x_n - \alpha)/(x_{n-1} - \alpha)|} \tag{2}$$

3 Some Modified Newton's Methods

Newton's method (NM) for computing the root α of the nonlinear equation (1) is to start with initial estimate x_0 sufficiently close to the root α and to use the one point iteration

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \tag{3}$$

where x_n is the n-th approximation of α. We may also view x_{n+1} as the root of the two-term Taylor expansion or linear model of f about x_n,[5],

$$M(x) = f(x_n) + f'(x_n)(x - x_n).$$

By integrating by part this local model can be viewed as the following obvious identity,[2],

$$f(x) = f(x_n) + \int_{x_n}^{x} f'(s)ds \tag{4}$$

Newton approximates $\int_{x_n}^{x} f'(s)ds$ in (4) using the left Riemann-sum for one interval, resulted in

$$\int_{x_n}^{x} f'(s)ds \approx f'(x_n)(x - x_n). \tag{5}$$

On substituting (5) into (4), setting $f(x) = 0$ rearranging the terms of the resulting equation, we end up with the equation (3).

Weerakoon and Fernando [7] approximate the indefinite integral involved in (4) by the trapezoidal rule,

$$\int_{x_n}^{x} f'(s)ds \approx \Big(\frac{f'(x_n) + f'(x)}{2}\Big)(x - x_n),$$

and then by some algebra they end up with the following scheme (TNM)

$$x_{n+1} = x_n - \frac{2f(x_n)}{f'(x_n) + f'(x^*_{n+1})} \tag{6}$$

$$x^*_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \tag{7}$$

They prove that the scheme is a third order convergence. The way they choose x^*_{n+1} as in (7) was introduced for the first time by Wall in [8].

Now from the approximation error of trapezoidal rule and midpoint rule, that is

$$E_T := \int_a^b f(x)dx - \frac{h}{2}(f(a) + f(b))) = -\frac{(b-a)^3}{12} f''(\xi), \quad \xi \in (a,b) \text{and} f \in C^2[a,b],$$

and

$$E_M := \int_a^b f(x)dx - (b-a)f(\frac{a+b}{2}) = \frac{(b-a)^3}{24} f''(\xi), \quad \xi \in (a,b) \text{and} f \in C^2[a,b],$$

we see that the approximation error of the midpoint rule needs the same smoothness as the trapezoidal rule. By comparing the constant in front of the derivative of f, the absolute value of midpoint approximation error is slightly smaller than the trapezoidal rule. From this view we may use the midpoint rule to approximate the indefinite integral involved in (4), that is

$$\int_{x_n}^{x} f'(s)ds \approx (x - x_n) f'\Big(\frac{x + x_n}{2}\Big)$$

Then, by some algebra we propose the scheme (MNM)

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x^*_{n+1})}, \tag{8}$$

$$x^*_{n+1} = x_n - \frac{f(x_n)}{2f'(x_n)}. \tag{9}$$

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4 Asymptotic Error Analysis

Theorema 3 Let $f : D \to \mathbb{R}$ for an open interval D. Assume that f has sufficiently differentiable function on the interval D. If f has a simple root $\alpha \in D$ and x_0 is sufficiently close to α, then the MNM defined by (8) and (9) satisfies the following error equation:

$$e_{n+1} = (c_2^2 + \frac{3}{4}c_3)e_n^3 + \mathcal{O}(e_n^4),$$

where $e_n = x_n - \alpha$ and

$$c_j = \frac{f^{(j)}(\alpha)}{j!\, f'(\alpha)}, \quad j = 2, 3.$$

Proof: See Imran [4]

Theorema 4 Let $f : D \to \mathbb{R}$ for an open interval D. Assume that f has sufficiently differentiable function on the interval D, and f has a multiple root α of multiplicity $m > 1 \in D$. If x_0 is sufficiently close to α, then the MNM defined by (8) and (9) satisfies the following error equation:

$$e_{n+1} = \Big(1 - \frac{1}{mK}\Big)e_n - \Big(\frac{(1-m)}{2Km^3}c_2 e_n^2 + \mathcal{O}(e_n^3)\Big),$$

where $e_n = x_n - \alpha$, $c_j = \frac{f^{(j)}(\alpha)}{j!\, f'(\alpha)}$, for $j = 2, 3$, $K = 1$ for $m = 1$ and $K \approx \Big(1 - \frac{1}{2m}\Big)^{m-1}$ for $m > 1$.

Proof:
We follow the technical proof from [6]. Let α be a root of multiplicity m, (i.e.$f(\alpha) = f'(\alpha) = \cdots = f^{(m)}(\alpha) = 0$, and $f^{(m+1)}(\alpha) \neq 0$. We expand $f(x_n)$ about $x = \alpha$ using Taylor expansion, that is

$$\begin{aligned} f(x_n) &= \frac{f^{(m)}(\alpha)}{m!}(x_n - \alpha)^m + \frac{f^{(m+1)}(\alpha)}{(m+1)!}(x_n - \alpha)^{m+1} + \frac{f^{(m+2)}(\alpha)}{(m+2)!}(x_n - \alpha)^{m+2} \\ &\quad + \frac{f^{(m+3)}(\alpha)}{(m+3)!}(x_n - \alpha)^{m+3} + \mathcal{O}((x_n - \alpha)^{m+4}) \\ &= \frac{f^{(m)}(\alpha)}{m!}e_n^m(1 + c_2 e_n + c_3 e_n^2 + c_4 e_n^3 + \mathcal{O}(e_n^4)), \end{aligned} \tag{10}$$

where

$$c_j = \frac{f^{(m+j-1)}(\alpha)}{f^{(m)}(\alpha)(m+1)(m+2)\cdots(m+j-1)}, \quad j = 2, 3, 4.$$

Moreover we have

$$\begin{aligned} f'(x_n) &= \frac{f^{(m)}(\alpha)}{(m-1)!}(x_n - \alpha)^{m-1} + \frac{f^{(m+1)}(\alpha)}{m!}(x_n - \alpha)^m \\ &\quad + \frac{f^{(m+2)}(\alpha)}{(m+1)!}(x_n - \alpha)^{m+1} + \mathcal{O}((x_n - \alpha)^{m+2}) \\ &= \frac{f^{(m)}(\alpha)}{(m-1)!}e_n^{m-1}\Big(1 + \frac{(m+1)f^{(m+1)}(\alpha)}{f^{(m)}(\alpha)m(m+1)}e_n \\ &\qquad + \frac{(m+2)f^{(m+2)}(\alpha)}{f^{(m)}(\alpha)m(m+1)(m+2)}e_n^2 + \mathcal{O}(e_n^3)\Big) \\ &= \frac{f^{(m)}(\alpha)}{(m-1)!}e_n^{m-1}\Big(1 + \frac{(m+1)}{m}c_2 e_n + \frac{(m+2)}{m}c_3 e_n^2 + \mathcal{O}(e_n^3)\Big). \end{aligned} \tag{11}$$

Computing $f(x_n)/(2f'(x_n))$ from (10) and (11) and recalling

$$(1 + x)^{-1} = 1 - x + x^2 - x^3 + x^4 - x^5 + \cdots + \mathcal{O}(x^n)$$

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we obtain after some algebra

$$
\begin{aligned}
\frac{f(x_n)}{2f'(x_n)} &= \frac{1}{2}\frac{\frac{f^{(m)}(\alpha)}{m!}e_n^m(1+c_2e_n+c_3e_n^2+c_4e_n^3+\mathcal{O}(e_n^4))}{\frac{f^{(m)}(\alpha)}{(m-1)!}e_n^{m-1}\Big(1+\frac{(m+1)}{m}c_2e_n+\frac{(m+2)}{m}c_3e_n^2+\mathcal{O}(e_n^3)\Big)}\\
&= \frac{1}{2m}(e_n+c_2e_n^2+c_3e_n^3+\mathcal{O}(e_n^4))\\
&\quad\times\Big(1+\frac{(m+1)}{m}c_2e_n+\frac{(m+2)}{m}c_3e_n^2++\mathcal{O}(e_n^3)\Big)^{-1}\\
&= \frac{1}{2m}(e_n+c_2e_n^2+c_3e_n^3+\mathcal{O}(e_n^4))\\
&\quad\times\Big[1-\Big(1+\frac{1}{m}\Big)c_2e_n+\Big(-c_3-\frac{2c_3}{m}+\frac{c_2^2}{m^2}+c_2^2+\frac{2c_2^2}{m}\Big)e_n^2+\mathcal{O}(e_n^3)\Big]\\
&= \frac{1}{2m}\Big[e_n-\frac{c_2}{m}e_n^2+\mathcal{O}(e_n^3)\Big].
\end{aligned}
\tag{12}
$$

On substituting (12) into (8) and simplifying yields

$$
\begin{aligned}
x_{n+1}^* &= x_n-\frac{1}{2m}\Big[e_n-\frac{c_2}{m}e_n^2+\mathcal{O}(e_n^3)\Big]\\
&= \alpha+\Big[\Big(1-\frac{1}{2m}\Big)e_n+\frac{c_2}{2m^2}e_n^2+\mathcal{O}(e_n^3)\Big]
\end{aligned}
\tag{13}
$$

or

$$
x_{n+1}^*-\alpha=\Big(1-\frac{1}{2m}\Big)e_n+\frac{c_2}{2m^2}e_n^2+\mathcal{O}(e_n^3).
\tag{14}
$$

Expanding $f(x_{n+1}^*)$ about $x=\alpha$, noting (14) and simplifying, we end up with

$$
\begin{aligned}
f'(x_{n+1}^*) &= \frac{f^{(m)}(\alpha)}{(m-1)!}\Big(\Big(1-\frac{1}{2m}\Big)e_n+\frac{c_2}{2m^2}e_n^2+\mathcal{O}(e_n^3)\Big)^{m-1}\\
&\quad\times\Big[1+\frac{(m+1)}{m}c_2\Big(\Big(1-\frac{1}{2m}\Big)e_n+\frac{c_2}{2m^2}e_n^2+\mathcal{O}(e_n^3)\Big)\\
&\qquad+\frac{(m+2)}{m}c_3\Big(\Big(1-\frac{1}{2m}\Big)e_n+\frac{c_2}{2m^2}e_n^2+\mathcal{O}(e_n^3)\Big)^2+\mathcal{O}(e_n^3)\Big]\\
&= \frac{f^{(m)}(\alpha)}{(m-1)!}\Big(\Big(1-\frac{1}{2m}\Big)e_n+\frac{c_2}{2m^2}e_n^2+\mathcal{O}(e_n^3)\Big)^{m-1}\\
&\quad\times\Big[1+\Big(\frac{2m^2+m-1}{2m^2}\Big)c_2e_n\\
&\qquad+\Big(\frac{(2m+2)}{4m^3}c_2+\frac{(4m^3+4m^2-7m+2)}{4m^3}c_3\Big)e_n^2+\mathcal{O}(e_n^3)\Big]\\
&= \frac{f^{(m)}(\alpha)}{(m-1)!}Ke_n^{m-1}\Big[1+\Big(\frac{2m^2+m-1}{2m^2}\Big)c_2e_n\\
&\qquad+\Big(\frac{(2m+2)}{4m^3}c_2+\frac{(4m^3+4m^2-7m+2)}{4m^3}c_3\Big)e_n^2+\mathcal{O}(e_n^3)\Big],
\end{aligned}
\tag{15}
$$

where

$$
K=\Big(\Big(1-\frac{1}{2m}\Big)+\frac{c_2}{2m^2}e_n+\mathcal{O}(e_n^2)\Big)^{m-1}.
$$

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Furthermore, using (10) and (15), and recalling (4), we compute $f(x_n)/f'(x^*_{n+1})$, that is

$$\frac{f(x_n)}{f'(x^*_{n+1})} = \frac{\frac{f^{(m)}(\alpha)}{m!}e_n^m(1+c_2e_n+c_3e_n^2+c_4e_n^3+\mathcal{O}(e_n^4))}{\frac{f^{(m)}(\alpha)}{(m-1)!}Ke_n^{m-1}\left[1+\left(\frac{2m^2+m-1}{2m^2}\right)c_2e_n+\left(\frac{(2m+2)}{4m^3}c_2+\frac{(4m^3+4m^2-7m+2)}{4m^3}c_3\right)e_n^2+\mathcal{O}(e_n^3)\right]}$$

$$= \frac{1}{mK}\frac{(e_n+c_2e_n^2+c_3e_n^3+\mathcal{O}(e_n^4))}{\left[1+\left(\frac{2m^2+m-1}{2m^2}\right)c_2e_n+\left(\frac{(2m+2)}{4m^3}c_2+\frac{(4m^3+4m^2-7m+2)}{4m^3}c_3\right)e_n^2+\mathcal{O}(e_n^3)\right]}$$

$$= \frac{1}{mK}(e_n+c_2e_n^2+c_3e_n^3+\mathcal{O}(e_n^4))$$

$$\times\left[1+\left(\frac{2m^2+m-1}{2m^2}\right)c_2e_n+\left(\frac{(2m+2)}{4m^3}c_2+\frac{(4m^3+4m^2-7m+2)}{4m^3}c_3\right)e_n^2+\mathcal{O}(e_n^3)\right]^{-1}$$

$$= \frac{1}{mK}\left(e_n+\frac{(1-m)}{2m^2}c_2e_n^2+\mathcal{O}(e_n^3)\right). \tag{16}$$

On substituting (16) into (9), we obtain

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x^*_{n+1})}$$

$$e_{n+1} = \left(1-\frac{1}{mK}\right)e_n - \left(\frac{(1-m)}{2Km^3}c_2e_n^2+\mathcal{O}(e_n^3)\right)$$

This end the proof.

5 Numerical Simulations

The modified methods and Newton's method are tested using some functions and initial points, which have been used [7], [3] and [1]. We compute the computational order of convergence (COC) using formula (2). We stop the program using the following criteria

$$\frac{|x_{n+1}-x_n|}{|x_{n+1}|} < \epsilon,$$

$$|f(x_{n+1})| < \epsilon,$$

$$|x_{n+1}-\alpha| < \epsilon,$$

where $\epsilon = 2.22e-10$ and α is the root. All programs are written in Matlab 7 and run on Windows PC with Intel Processor at 2.4 GHz. The computational results for the case of simple roots are given in Table 1 and for the multiple roots are given in Table 2 .

Table 1: Comparisons of the number of iterations and the COC of the modified methods for the case of simple roots

$f(x)$	x_0	Number of Iterations			COC		
		NM	MNM	TNM	NM	MNM	TNM
$\cos(x)-x$	-1.0	8	6	3	ND	ND	ND
	1.7	4	3	3	2.006	ND	ND
	2.0	3	3	3	2.001	ND	2.996
	3.0	6	3	8	2.001	2.958	2.885
	4.0	29	4	6	ND	ND	2.978
$(x-1.0)^3-1$	2.5	5	3	3	1.998	2.985	2.979
	4.0	7	4	5	2.019	2.971	ND
	-0.5	15	5	15	1.999	ND	3.013
	-1.0	10	5	7	ND	2.993	3.018
	-2.0	10	6	8	2.014	3.013	ND
$xe^{(x^2)}-\sin^2(x)+$	-3.0	13	18	9	2.000	3.019	ND
$3\cos(x)+5.0$	1.2	100+	38	20	1.002	ND	ND
$e^{(x^2+7x-30)}-1$	3.3	8	5	6	2.000	2.991	ND
	3.5	11	7	8	2.000	2.639	ND

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Table 2: Comparisons of the number of iterations and the COC of the modified methods for the case of multiple roots

$f(x)$	x_0	Number of Iterations			COC		
		NM	MNM	TNM	NM	MNM	TNM
$x^2-2.22x+1.2321$	-1.0	18	11	11	1.000	1.000	1.000
	0.6	16	10	10	1.000	1.000	1.000
	2.2	17	11	11	1.000	1.000	1.000
	-10.0	20	13	13	1.000	1.000	1.000
$x^4-5.4x^3+10.56x^2-8.954x+2.7951$	0.6	18	11	12	1.000	1.000	1.000
	0.8	16	10	11	1.000	1.000	1.000
	1.4	15	10	10	1.000	1.000	1.000
	1.8	18	13	11	1.000	1.000	1.000
$x^3-5.56x^2+9.1389x-4.68999$	0.0	18	11	12	1.000	1.000	1.000
	0.5	17	11	11	1.000	1.000	1.000
	1.5	15	10	10	1.000	1.000	1.000
	-2.0	20	13	13	1.000	1.000	1.000
$x^4-8x^3+24x^2-32x+16$	-2.5	25	16	17	1.000	1.000	1.000
	0.0	22	14	15	1.000	1.000	1.000
	4.0	22	14	15	1.000	1.000	1.000
	10.0	27	17	18	1.000	1.000	1.000

From the Table 1, we see that some of the COCs is not defined (ND). This occurs because division by zero while applying the formula (2). The COC of all methods for the case of simple roots matches the theoretical results, both for Newton's method and the modified methods. In general, the number of iterations produced by applying the MNM is fewer than the other methods, except at the starting point $x_0 = -3.0$ and $x_0 = 1.2$ for $f(x) = xe^{(x^2)} - \sin^2(x) + 3\cos(x) + 5.0$, where MNM method needs 18 and 38 number of iterations respectively.

From the Table 2, we see that the COC for the all methods for the case of multiple roots is one. This matches the theoretical result as stated in the Theorem 4 for the MNM. In general, the number of iterations produced by applying the MNM is fewer than the other methods, except at the starting point $x_0 = 1.8$ for $f(x) = x^4 - 5.4x^3 + 10.56x^2 - 8.954x + 2.7951$, where MNM method needs 13 number of iterations. In all experiment for the multiple roots, the iteration ends not because of test of error given fulfilled, but because the value of function satisfies the second stopping criteria. This matter represents especial constraint in approximating multiple roots.

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