SOOCHOW JOURNAL OF MATHEMATICS
Volume 27, No. 3, pp. 321-329, July 2001

# ON FINITE DIMENSIONAL 2-NORMED SPACES

BY

HENDRA GUNAWAN AND MASHADI

**Abstract**. In this note, we shall study finite dimensional 2-normed spaces and show that their topology can be fully described by using a certain norm derived from the 2-norm. Moreover, we show that a 2-Banach space is a Banach space and use this fact to prove the Fixed-Point Theorem. Our result extends to some infinite dimensional 2-normed spaces.

## 1. Introduction

The concepts of 2-metric spaces and 2-normed spaces were intially introduced by Gähler [ 5, 6, 7] in 1960's. Since then, many researchers have studied these two spaces and obtained various results, see for instance [1, 2, 9, 10, 11, 12].

One well-known fact about 2-normed spaces is that they are actually normed spaces. In this note, we shall study finite dimensional 2-normed spaces and show that their topology can be fully described by using a certain norm derived from the 2-norm. Moreover, we shall show that a 2-Banach space is a Banach space and use this fact to prove results such as the Fixed-Point Theorem. We also indicate that our result extends to some infinite dimensional 2-normed spaces.

Let $X$ be a real vector space of dimension $d$, where $2 \leq d < \infty$. A 2-*norm* on $X$ is a function $\|\cdot,\cdot\| : X \times X \to \mathbf{R}$ which satisfies the following four conditions:

(i) $\|\mathbf{x},\mathbf{y}\| = 0$ if and only if $\mathbf{x}$ and $\mathbf{y}$ are linearly dependent;
(ii) $\|\mathbf{x},\mathbf{y}\| = \|\mathbf{y},\mathbf{x}\|$;
(iii) $\|\mathbf{x},\alpha\mathbf{y}\| = |\alpha|\,\|\mathbf{x},\mathbf{y}\|,\ \alpha \in \mathbf{R}$;

Received March 16, 2000; revised September 19, 2000.
AMS Subject Classification. 46B20, 46B99, 47H10, 54H25.
*Key words.* 2-normed spaces, normed spaces, fixed-point theorem.

(iv) $\|\mathbf{x}, \mathbf{y}+\mathbf{z}\| \leq \|\mathbf{x}, \mathbf{y}\| + \|\mathbf{x}, \mathbf{z}\|$.
The pair $(X, \|\cdot,\cdot\|)$ is then called a 2-*normed space*.

A standard example of a 2-normed space is $\mathbf{R}^2$ equipped with the following 2-norm

$$\|\mathbf{x}, \mathbf{y}\| \ := \ \text{the area of the triangle having vertices } \mathbf{0},\ \mathbf{x} \text{ and } \mathbf{y}.$$

Observe that in any 2-normed space $(X, \|\cdot,\cdot\|)$ we have $\|\mathbf{x}, \mathbf{y}\| \geq 0$ and $\|\mathbf{x}, \mathbf{y} + \alpha\mathbf{x}\| = \|\mathbf{x}, \mathbf{y}\|$ for all $\mathbf{x}, \mathbf{y} \in X$ and $\alpha \in \mathbf{R}$. Also, if $\mathbf{x}, \mathbf{y}$ and $\mathbf{z}$ are linearly dependent (this happens, for instance, when $d = 2$), then $\|\mathbf{x}, \mathbf{y}+\mathbf{z}\| = \|\mathbf{x}, \mathbf{y}\| + \|\mathbf{x}, \mathbf{z}\|$ or $\|\mathbf{x}, \mathbf{y}-\mathbf{z}\| = \|\mathbf{x}, \mathbf{y}\| + \|\mathbf{x}, \mathbf{z}\|$.

Given a 2-normed space $(X, \|\cdot,\cdot\|)$, one can derive a topology for it via the following definition of the limit of a sequence: A sequence $(\mathbf{x}_n)$ in $X$ is said to be *convergent* to $\mathbf{x}$ in $X$ if $\lim\limits_{n\to\infty} \|\mathbf{x}_n - \mathbf{x}, \mathbf{y}\| = 0$ for every $\mathbf{y} \in X$. In such a case, we write $\lim\limits_{n\to\infty} \mathbf{x}_n := \mathbf{x}$ and call $\mathbf{x}$ the *limit* of $(\mathbf{x}_n)$.

The uniqueness of the limit of a convergent sequence can be verified as follows. Suppose $(\mathbf{x}_n)$ is convergent to two distinct limits $\mathbf{x}$ and $\mathbf{y}$ in $X$. Choose $\mathbf{z} \in X$ such that $\|\mathbf{x}-\mathbf{y}, \mathbf{z}\| \neq 0$ and take $N \in \mathbf{N}$ sufficiently large such that $\|\mathbf{x}_N - \mathbf{x}, \mathbf{z}\| < \frac{1}{2}\|\mathbf{x}-\mathbf{y}, \mathbf{z}\|$ and $\|\mathbf{x}_N - \mathbf{y}, \mathbf{z}\| < \frac{1}{2}\|\mathbf{x}-\mathbf{y}, \mathbf{z}\|$ simultaneously. Then, by the triangle inequality, we obtain

$$\begin{aligned} \|\mathbf{x}-\mathbf{y}, \mathbf{z}\| &\leq \|\mathbf{x}-\mathbf{x}_N, \mathbf{z}\| + \|\mathbf{x}_N - \mathbf{y}, \mathbf{z}\| \\ &< \frac{1}{2}\|\mathbf{x}-\mathbf{y}, \mathbf{z}\| + \frac{1}{2}\|\mathbf{x}-\mathbf{y}, \mathbf{z}\| = \|\mathbf{x}-\mathbf{y}, \mathbf{z}\|, \end{aligned}$$

which is absurd. Hence, whenever it exists, $\lim\limits_{n\to\infty} \mathbf{x}_n$ must be unique.

Now that we have defined convergent sequences in a 2-normed space, it is natural to inquire how we can define balls there so that we can fully describe its topology by using these balls. In the next section, we show how to do this for finite dimensional case. Moreover, we show that a 2-Banach space is a Banach space and use this fact to prove the Fixed-Point Theorem. In the last section we indicate that our result extends to some infinite dimensional 2-normed spaces.

## 2. Main Results

Suppose hereafter that $(X, \|\cdot,\cdot\|)$ is a 2-normed space. Recall that we assume $X$ to have dimension $d$, where $2 \leq d < \infty$, unless otherwise stated. Fix $\{\mathbf{u}_1, \ldots, \mathbf{u}_d\}$ to be a basis for $X$. Then we have the following:

**Lemma 2.1.** *A sequence* $(\mathbf{x}_n)$ *in* $X$ *is convergent to* $\mathbf{x}$ *in* $X$ *if and only if* $\lim_{n\to\infty} \|\mathbf{x}_n - \mathbf{x}, \mathbf{u}_i\| = 0$ *for every* $i = 1, \dots, d$.

**Proof.** It suffices to prove that if $\lim_{n\to\infty} \|\mathbf{x}_n - \mathbf{x}, \mathbf{u}_i\| = 0$ for every $i = 1, \dots, d$, then we have $\lim_{n\to\infty} \|\mathbf{x}_n - \mathbf{x}, \mathbf{y}\| = 0$ for every $\mathbf{y} \in X$. But this is clear since every $\mathbf{y} \in X$ can be written as $\mathbf{y} = \alpha_1\mathbf{u}_1 + \ldots + \alpha_d\mathbf{u}_d$ for some $\alpha_1, \dots, \alpha_d \in \mathbf{R}$, and by the triangle inequality we have

$$\|\mathbf{x}_n - \mathbf{x}, \mathbf{y}\| \leq |\alpha_1| \, \|\mathbf{x}_n - \mathbf{x}, \mathbf{u}_1\| + \cdots + |\alpha_d| \, \|\mathbf{x}_n - \mathbf{x}, \mathbf{u}_d\|$$

for all $n \in \mathbf{N}$.

Following Lemma 2.1, we have:

**Lemma 2.2.** *A sequence* $(\mathbf{x}_n)$ *in* $X$ *is convergent to* $\mathbf{x}$ *in* $X$ *if and only if* $\lim_{n\to\infty} \max\{\|\mathbf{x}_n - \mathbf{x}, \mathbf{u}_i\| \, : \, i = 1, \dots, d\} = 0$.

This simple fact enlightens us to define a norm on $X$ as follows: With respect to the basis $\{\mathbf{u}_1, \dots, \mathbf{u}_d\}$, we can define a norm on $X$, which we shall denote it by $\|\cdot\|_\infty$, by

$$\|\mathbf{x}\|_\infty := \max\{\|\mathbf{x}, \mathbf{u}_i\| \, : \, i = 1, \dots, d\}.$$

Indeed, one may observe that: (i) $\|\mathbf{x}\|_\infty = 0$ if and only if $\mathbf{x} = \mathbf{0}$, (ii) $\|\alpha\mathbf{x}\|_\infty = |\alpha| \, \|\mathbf{x}\|_\infty$, and (iii) $\|\mathbf{x} + \mathbf{y}\|_\infty \leq \|\mathbf{x}\|_\infty + \|\mathbf{y}\|_\infty$ for all $\mathbf{x}, \mathbf{y} \in X$ and $\alpha \in \mathbf{R}$. (One might like to compare our norm to the one offered by Gähler in [5] or [6].)

Note that for $1 \leq p \leq \infty$ in general, we can define the function $\|\cdot\|_p$ on $X$ by

$$\|\mathbf{x}\|_p := \Big\{\sum_{i=1}^{d} \|\mathbf{x}, \mathbf{u}_i\|^p\Big\}^{1/p},$$

and check that it also defines a norm on $X$. But, since $X$ is finite dimensional, all these norms are equivalent, and so throughout this section we shall only work with $\|\cdot\|_\infty$, unless otherwise required.

Note also that the choice of the basis here is not essential. If we choose another basis for $X$, say $\{\mathbf{v}_1, \dots, \mathbf{v}_d\}$, and define the norm $\|\cdot\|_\infty$ with respect to it, then the resulting norm will be equivalent to the one defined with respect to $\{\mathbf{u}_1, \dots, \mathbf{u}_d\}$.

Using the derived norm $\|\cdot\|_\infty$, Lemma 2.2 now reads:

**Lemma 2.2$^\dagger$.** *A sequence* $(\mathbf{x}_n)$ *in* $X$ *is convergent to* $\mathbf{x}$ *in* $X$ *if and only if* $\lim_{n\to\infty} \|\mathbf{x}_n - \mathbf{x}\|_\infty = 0$.

Associated to the derived norm $\|\cdot\|_\infty$, we can define the (open) balls $B_{\{\mathbf{u}_1,..,\mathbf{u}_d\}}(\mathbf{x}, r)$ centered at $\mathbf{x}$ having radius $r$ by

$$B_{\{\mathbf{u}_1,\ldots,\mathbf{u}_d\}}(\mathbf{x}, r) := \{\mathbf{y} : \|\mathbf{x}-\mathbf{y}\|_\infty < r\}.$$

Using these balls, Lemma 2.2$^\dagger$ becomes:

**Lemma 2.2$^\ddagger$.** *A sequence* $(\mathbf{x}_n)$ *is convergent to* $\mathbf{x}$ *in* $X$ *if and only if* $\forall\, \epsilon > 0\ \exists\, N \in \mathbf{N}$ *such that* $n \geq N \;\Rightarrow\; \mathbf{x}_n \in B_{\{\mathbf{u}_1,\ldots,\mathbf{u}_d\}}(\mathbf{x}, \epsilon)$.

Summarizing all these results, we have:

**Theorem 2.3.** *Any finite dimensional* 2*-normed space is a normed space and its topology agrees with that generated by the derived norm* $\|\cdot\|_\infty$.

As we shall see below, many results in a normed space can be verified analogously in a 2-normed space by using the derived norm $\|\cdot\|_\infty$ and/or its associated balls. But first let us see some examples.

**Example 2.4.** Let $X = \mathbf{R}^2$ be equipped with the 2-norm $\|\mathbf{x}, \mathbf{y}\| :=$ the area of the parallelogram spanned by the vectors $\mathbf{x}$ and $\mathbf{y}$ (= twice the area of the triangle having vertices $\mathbf{0}$, $\mathbf{x}$ and $\mathbf{y}$), which may be given explicitly by the formula

$$\|\mathbf{x}, \mathbf{y}\| = |x_1y_2 - x_2y_1|, \quad \mathbf{x} = (x_1, x_2),\ \mathbf{y} = (y_1, y_2).$$

Take the standard basis $\{\mathbf{i}, \mathbf{j}\}$ for $\mathbf{R}^2$. Then, $\|\mathbf{x}, \mathbf{i}\| = |x_2|$ and $\|\mathbf{x}, \mathbf{j}\| = |x_1|$, and so the derived norm $\|\cdot\|_\infty$ with respect to $\{\mathbf{i}, \mathbf{j}\}$ is

$$\|\mathbf{x}\|_\infty = \max\{|x_1|, |x_2|\}, \quad \mathbf{x} = (x_1, x_2).$$

Thus the derived norm $\|\cdot\|_\infty$ here is exactly the same as the uniform norm on $\mathbf{R}^2$. Accordingly, the ball $B_{\{\mathbf{i},\mathbf{j}\}}(\mathbf{x}, r)$ is a square centered at $\mathbf{x}$ having "radius" $r$. Since the derived norm is equivalent to the Euclidean norm on $\mathbf{R}^2$, we conclude that $\mathbf{R}^2$ equipped with the above 2-norm is nothing but the Euclidean plane.

More generally, we have:

**Fact 2.5.** *The* 2*-normed space* $(\mathbf{R}^d, \|\cdot,\cdot\|)$ *where* $\|\mathbf{x},\mathbf{y}\|$ := *the area of the parallelogram spanned by* $\mathbf{x}$ *and* $\mathbf{y}$ *is a normed space whose norm is equivalent to the Euclidean's.*

**Proof.** For every $\mathbf{x} = (x_1, \dots, x_d)$ and $\mathbf{y} = (y_1, \dots, y_d) \in \mathbf{R}^d$, the 2-norm $\|\mathbf{x},\mathbf{y}\|$ may be given explicitly by the formula

$$\|\mathbf{x},\mathbf{y}\| = \left\{ \Big(\sum_{i=1}^d x_i^2\Big)\Big(\sum_{j=1}^d y_j^2\Big) - \Big(\sum_{i=1}^d x_i y_i\Big)^2 \right\}^{1/2}.$$

Let $\{\mathbf{e}_1, \dots, \mathbf{e}_d\}$ be the standard basis for $\mathbf{R}^d$. Then, for each $j = 1, \dots, d$, we have

$$\|\mathbf{x},\mathbf{e}_j\| = \left\{ \Big(\sum_{i=1}^d x_i^2\Big) - x_j^2 \right\}^{1/2},$$

and so the derived norm $\|\cdot\|_\infty$ with respect to $\{\mathbf{e}_1, \dots, \mathbf{e}_d\}$ is

$$\|\mathbf{x}\|_\infty = \max\left\{ \left\{ \Big(\sum_{i=1}^d x_i^2\Big) - x_j^2 \right\}^{1/2} : j = 1, \dots, d \right\}.$$

Now let $\|\cdot\|_{\mathrm{E}}$ denote the Euclidean norm on $\mathbf{R}^d$. Then it is easy to check that

$$\|\mathbf{x}\|_\infty \le \|\mathbf{x}\|_{\mathrm{E}} \le \sqrt{2}\|\mathbf{x}\|_\infty$$

for all $\mathbf{x} \in \mathbf{R}^d$, confirming that the derived norm is equivalent to the Euclidean's.

**Remark.** For the above 2-normed space $(\mathbf{R}^d, \|\cdot,\cdot\|)$, it is worth observing that $\|\mathbf{x}\|_2^2 = \sum_{j=1}^d \|\mathbf{x},\mathbf{e}_j\|^2 = (d-1)\sum_{i=1}^d x_i^2$, that is, $\|\cdot\|_2$ is a multiple of the Euclidean norm.

We are now about to prove the Fixed Point Theorem for finite dimensional 2-Banach spaces. (Recall that $(X, \|\cdot,\cdot\|)$ is a 2-Banach space if every Cauchy sequence in $X$, that is any sequence $(\mathbf{x}_n)$ in $X$ such that $\lim_{m,n\to\infty} \|\mathbf{x}_m - \mathbf{x}_n, \mathbf{y}\| = 0$ for every $\mathbf{y} \in X$, is convergent to some $\mathbf{x}$ in $X$.) But first we need the following:

**Lemma 2.6.** $(X, \|\cdot,\cdot\|)$ *is a* 2*-Banach space if and only if* $(X, \|\cdot\|_\infty)$ *is a Banach space.*

**Proof.** Since, by Lemma 2.2, the convergence in the 2-norm is equivalent to that in the derived norm, it suffices to show that $(\mathbf{x}_n)$ is Cauchy with respect to the 2-norm if and only if it is Cauchy with respect to the derived norm. But this is clear since $(\mathbf{x}_n)$ is Cauchy with respect to the 2-norm if and only if $\lim\limits_{m,n\to\infty} \|\mathbf{x}_m - \mathbf{x}_n, \mathbf{y}\| = 0$ for every $\mathbf{y} \in X$, if and only if $\lim\limits_{m,n\to\infty} \|\mathbf{x}_m - \mathbf{x}_n, \mathbf{u}_i\| = 0$ for every $i = 1, \ldots, d$, if and only if $\lim\limits_{m,n\to\infty} \|\mathbf{x}_m - \mathbf{x}_n\|_\infty = 0$, if and only if $(\mathbf{x}_n)$ is Cauchy with respect to the derived norm.

Following Lemma 2.6, we have:

**Corollary 2.7.**(Fixed Point Theorem) *Suppose* $(X, \|\cdot,\cdot\|)$ *is a* 2-*Banach space. Let* $T$ *be a self-mapping of* $X$ *such that*

$$\|T\mathbf{x} - T\mathbf{y}, \mathbf{z}\| \le k\|\mathbf{x} - \mathbf{y}, \mathbf{z}\|$$

*for all* $\mathbf{x}, \mathbf{y}, \mathbf{z}$ *in* $X$*, where* $k$ *is a constant in* $(0,1)$*. Then* $T$ *has a unique fixed point in* $X$*.*

**Proof.** With respect to the derived norm $\|\cdot\|_\infty$, the mapping $T$ satisfies

$$\|T\mathbf{x} - T\mathbf{y}\|_\infty \le k\|\mathbf{x} - \mathbf{y}\|_\infty$$

for all $\mathbf{x}, \mathbf{y}$ in $X$. Since $(X, \|\cdot\|_\infty)$ is also a Banach space, we conclude by the Fixed Point Theorem for Banach spaces that $T$ has a unique fixed point in $X$.

## 3. Further Results

We shall now show that our result extends to any separable inner-product space $X$ (which may be of infinite dimension) equipped with the standard 2-norm

$$\|\mathbf{x}, \mathbf{y}\| := \left\{\|\mathbf{x}\|^2\|\mathbf{y}\|^2 - \langle\mathbf{x}, \mathbf{y}\rangle^2\right\}^{1/2},$$

where $\langle\cdot,\cdot\rangle$ denotes the inner-product and $\|\mathbf{x}\| := \langle\mathbf{x}, \mathbf{x}\rangle^{1/2}$ is its induced norm on $X$ (see [8] for discussion on the triangle inequality for this 2-norm).

First we verify an analogue of Fact 2.5. Let $(\mathbf{e}_i)$, indexed by a countable set $I \supseteq \{1, 2\}$, be an orthonormal basis for $X$. Then, for each $i \in I$, we have

$\|\mathbf{x}, \mathbf{e}_i\| = \left\{\|\mathbf{x}\|^2 - \langle \mathbf{x}, \mathbf{e}_i\rangle^2\right\}^{1/2} \leq \|\mathbf{x}\|$. Hence we may define the derived norm $\|\cdot\|_\infty$ with respect to $(\mathbf{e}_i)$ by

$$\|\mathbf{x}\|_\infty := \sup\{\|\mathbf{x}, \mathbf{e}_i\| \,:\, i \in I\}.$$

Clearly $\|\mathbf{x}\|_\infty \leq \|\mathbf{x}\|$ for all $\mathbf{x} \in X$. Conversely, using Bessel's inequality, we have

$$\|\mathbf{x}\|^2 \leq \|\mathbf{x}\|^2 - \langle \mathbf{x}, \mathbf{e}_1\rangle^2 + \|\mathbf{x}\|^2 - \langle \mathbf{x}, \mathbf{e}_2\rangle^2 = \|\mathbf{x}, \mathbf{e}_1\|^2 + \|\mathbf{x}, \mathbf{e}_2\|^2 \leq 2\|\mathbf{x}\|_\infty^2,$$

and hence $\|\mathbf{x}\| \leq \sqrt{2}\|\mathbf{x}\|_\infty$ for all $\mathbf{x} \in X$. This shows that $\|\cdot\|_\infty$ is equivalent to the already existing norm $\|\cdot\|$ on $X$.

Further, we see that the analogue of Lemma 2.1 is still valid. That is, a sequence $(\mathbf{x}_n)$ in $X$ is convergent to $\mathbf{x}$ in $X$ if and only if $\lim_{n\to\infty} \|\mathbf{x}_n - \mathbf{x}, \mathbf{e}_i\| = 0$ for every $i \in I$. Indeed, given $\lim_{n\to\infty} \|\mathbf{x}_n - \mathbf{x}, \mathbf{e}_i\| = 0$ for every $i \in I$, we can show that $\lim_{n\to\infty} \|\mathbf{x}_n - \mathbf{x}, \mathbf{y}\| = 0$ for every $\mathbf{y} \in X$. Observe that $\|\mathbf{x}_n - \mathbf{x}, \mathbf{y}\| \leq \|\mathbf{x}_n - \mathbf{x}\|\,\|\mathbf{y}\|$ for every $\mathbf{y} \in X$. Again, by Bessel's inequality, we have

$$\|\mathbf{x}_n - \mathbf{x}\|^2 \leq \|\mathbf{x}_n - \mathbf{x}, \mathbf{e}_1\|^2 + \|\mathbf{x}_n - \mathbf{x}, \mathbf{e}_2\|^2$$

for all $n \in \mathbf{N}$. Hence $\lim_{n\to\infty} \|\mathbf{x}_n - \mathbf{x}\| = 0$, and therefore $\lim_{n\to\infty} \|\mathbf{x}_n - \mathbf{x}, \mathbf{y}\| = 0$ for every $\mathbf{y} \in X$.

As it turns out, we obtain the following result:

**Fact 3.1.** *A sequence $(\mathbf{x}_n)$ in $X$ is convergent to $\mathbf{x}$ in $X$ if and only if $\lim_{n\to\infty} \|\mathbf{x}_n - \mathbf{x}, \mathbf{e}_i\| = 0$ for $i = 1$ and $2$ only.*

Accordingly, with respect to $\{\mathbf{e}_1, \mathbf{e}_2\}$, we can define a simpler norm $\|\cdot\|_2$ on $X$ by

$$\|\mathbf{x}\|_2 := \{\|\mathbf{x}, \mathbf{e}_1\|^2 + \|\mathbf{x}, \mathbf{e}_2\|^2\}^{1/2}.$$

This is not surprising since we can in general use two linearly dependent vectors to define a norm on any 2-normed space. What is remarkable here is that the topology generated by $\|\cdot\|_2$ agrees with that generated by the 2-norm. Observe that

$$\|\mathbf{x}\|_\infty \leq \|\mathbf{x}\| \leq \|\mathbf{x}\|_2 \leq \sqrt{2}\,\|\mathbf{x}\|_\infty$$

for all $\mathbf{x} \in X$. This confirms that $\|\cdot\|_2$ and $\|\cdot\|_\infty$ are equivalent, and both of them are equivalent to the already existing norm $\|\cdot\|$ on $X$. Hence the analogue of Lemma 2.2 (and its variants), Lemma 2.6 and Corollary 2.7 are still valid for $X$, whether it is equipped with $\|\cdot\|_\infty$, $\|\cdot\|_2$ or $\|\cdot\|$.

**Concluding Remark.** One might realize that the 2-norm discussed in the preceding section satisfies the parallelogram identity

$$\|\mathbf{x}+\mathbf{y},\mathbf{z}\|^2 + \|\mathbf{x}-\mathbf{y},\mathbf{z}\|^2 = 2(\|\mathbf{x},\mathbf{z}\|^2 + \|\mathbf{y},\mathbf{z}\|^2).$$

Another remarkable fact about the derived 2-norm $\|\cdot\|_2$ is that it inherits the identity

$$\|\mathbf{x}+\mathbf{y}\|_2^2 + \|\mathbf{x}-\mathbf{y}\|_2^2 = 2(\|\mathbf{x}\|_2^2 + \|\mathbf{y}\|_2^2).$$

This observation tells us that we can also derive an inner product from a 2-inner product (see [3] and [4] for the concept of 2-inner product spaces) and conclude that any 2-inner product space is actually an inner product space. Further results in this direction will appear elsewhere.

## Acknowledgment

The authors would like to thank the anonymous referees for their useful comments and suggestions.

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Department of Mathematics, Bandung Institute of Technology, Bandung 40132, Indonesia.

Department of Mathematics, University of Riau, Pekanbaru 28293, Indonesia.