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SEMI-EXCIRCLE OF QUADRILATERAL

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Abstract

Suppose that *ABCD* is a quadrilateral. Then there will be 4 pieces of semi-excircle on the quadrilateral. In this paper, we will discuss how to determine the length of the radii of the semi-excircle, but previously we will also be discussed how to determine the length of 4 new sides formed from the extension of the sides on $\square ABCD$ which has no parallel sides.

1. Introduction

On a triangle always can be formed incircle and excircle [1-3, 7, 15-17], but on any quadrilateral not necessarily can be formed incirle and excircle. Anyone have incircle but do not have excircle and there has excircle but do not have incircle [1, 3, 4, 7]. Especially for excircle already discussed in [10-13], but in [10, 12] only discusses about a quadrilateral has an excircle. For any convex quadrilateral actually made by [12] is not an excircle, because only offend a side of the rectangle and the other of two sides of extension, as in Figure 1.1.

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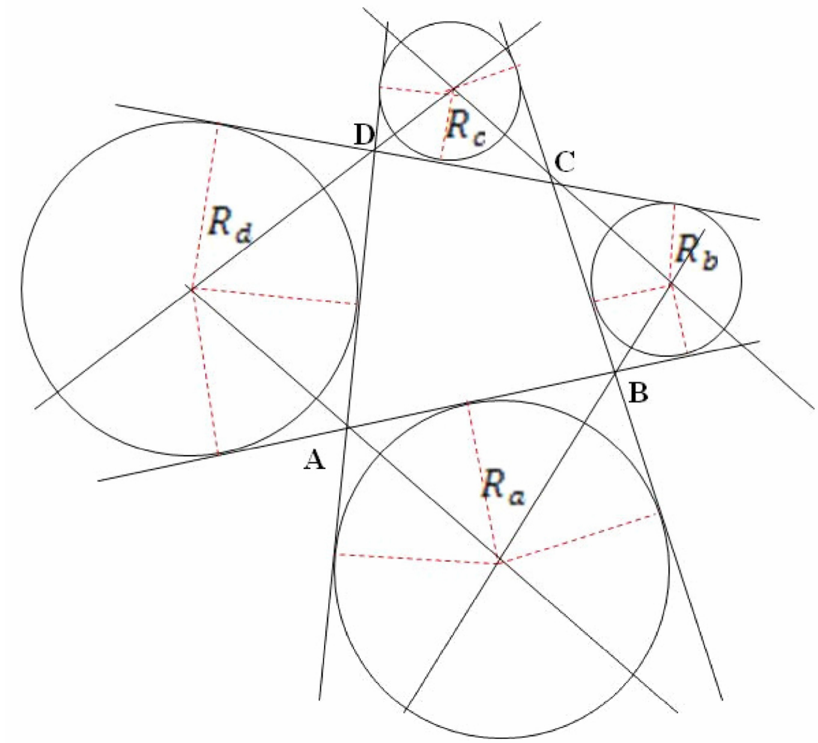

Figure 1.1

In this paper, a circle centered on the E, F, G and H (see Figure 1.2) everything is called *semi-excircle of quadrilateral ABCD*. We called semi-excircle, for example circle centered in E, only offensive side $a = AB$ and extension of the CB and DA, as well as a circle centered at F, G and H whereas R_a, R_b, R_c and R_d each is the radius of the semi-exircle centered in E, F, G and H. In [9-13] is called *tangential*. In [10, 12] is not explicitly how long the fingers, but an association that applies is $R_a \cdot R_c = R_b \cdot R_d$.

If $ABCD$ is a quadrilateral that there should be no parallel side, then there will be two pairs of intersecting side (details see Figure 1.2). Suppose that $P = AB \cap DC$ and $Q = AD \cap BC$. Then it needs to be discussed, how long of side $\alpha = BP$, $\beta = CP$, $\gamma = CQ$ and $\delta = DQ$. If $\square ABCD$ is cyclic quadrilateral, to calculate the length of each has been discussed on [7, 15, 17] for example $\alpha = \frac{\rho \cdot AC}{R}$, with ρ is a radius of the excircle ΔADC and R radius is a cyclic quadrilateral. The question is what if $\square ABCD$ not a cyclic quadrilateral. It is necessary to set the length of α, β, γ and δ and the length of radii of each of the semi-excircle.

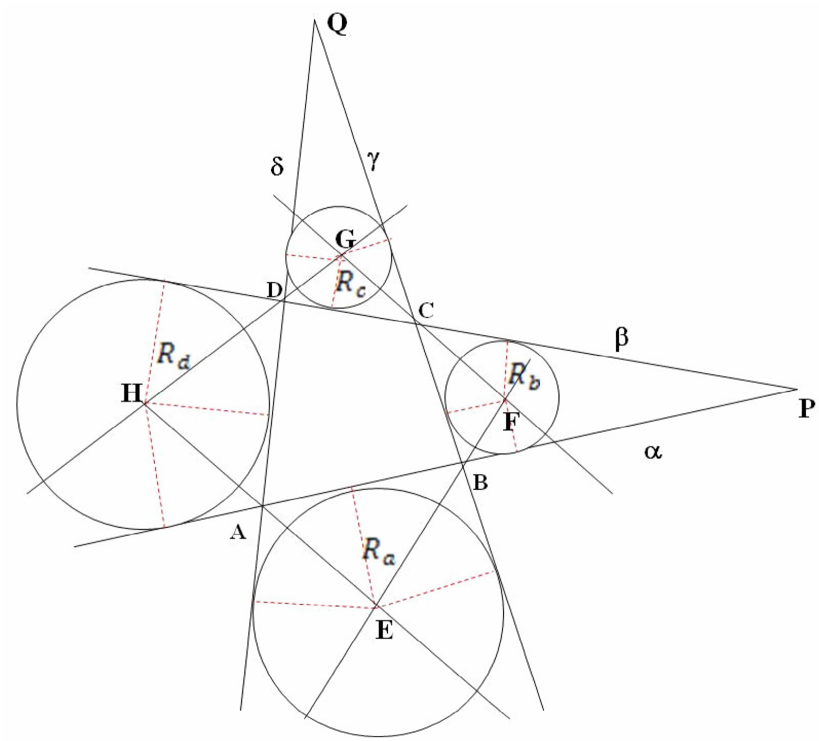

Figure 1.2

2. Semi-Gergonne and Long of New Side

If on any triangle, we always have 3 pieces of outer of Gergonne point. But in any quadrilateral has not been able to construct outer of Gergonne point, because it will not be possible to construct excircle for rectangular, except $\square ABCD$ has an excircle, see [8, 10, 11, 13]. Because that always can be constructed is the offending one side and the extension of the other two sides (on the condition that there should be no parallel sides), so that, each 2 pieces of which are incircle of ΔBCP and ΔCDQ were two others are excircle of ΔAPD and ΔBQA. But his fourth was outside of $\square ABCD$, so that lines drawn to each point of tangency is called the semi-

Gergonne of □*ABCD*. As for the four Gergonne points is like Figure 2.1. Noteworthy that I_a and I_b are the center point of the incircle ΔBCP and ΔCDQ, while I_c and I_d each of which is center point of excircle from ΔAPD and ΔBQA. While G_{e_1}, G_{e_2}, G_{e_3} and G_{e_4} are a Semi-Gergonne point on □*ABCD*.

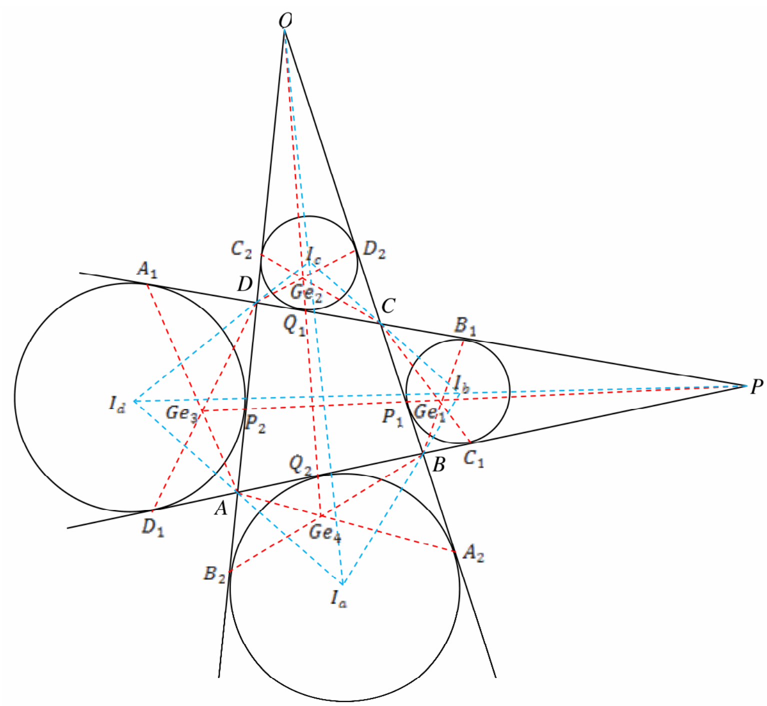

Figure 2.1

Noteworthy that impossible the incenter can be the same as semi-Gergonne point, except the triangle that formed is an equilateral triangle. On □*ABCD* convex, if each side is extended, then there will be the side that meets in one point, then forms some new sides. In Figure 2.1, there is no parallel of quadrilateral side. Four sides are different lengths between each other.

On Figure 2.1, can always be shown that $\Delta ADP \sim \Delta CBP$, it follows that

$$\frac{d}{b} = \frac{a+\alpha}{\beta}$$

$$= \frac{c+\beta}{\alpha}.$$

So

$$\beta = \frac{ab + b\alpha}{d}$$

and

$$d\alpha = bc + b\beta$$

which result in

$$d\alpha = bc + b\left(\frac{ab + b\alpha}{d}\right),$$

so that

$$\alpha = \frac{bcd + ab^2}{d^2 - b^2}$$

and

$$\beta = \frac{abd + cb^2}{d^2 - b^2}.$$

In another part, since $\Delta ABQ \sim \Delta CDQ$, so that

$$\frac{a}{c} = \frac{d + \delta}{\gamma}$$

$$= \frac{b + \gamma}{\delta}.$$

So

$$\gamma = \frac{cd + c\delta}{a}$$

and

$$a\delta = bc + c\gamma$$

which produce

$$\delta = \frac{abc + dc^2}{a^2 - c^2}$$

and

$$\gamma = \frac{acd + bc^2}{a^2 - c^2}.$$

Obviously the above conditions only apply to the value of $d > b$ and $a > c$, but if otherwise if $d < b$ and $a < c$, then the value of α, β, γ and δ will be negative. It is not possible, because of the long side is not possible negative. Therefore, the absolute value is taken. So the length of the side of α, β, γ and δ becomes

$$BP = \alpha$$

$$= \frac{bcd + ab^2}{|d^2 - b^2|},$$

$$CP = \beta$$

$$= \frac{abd + cb^2}{|d^2 - b^2|},$$

$$CQ = \gamma$$

$$= \frac{acd + bc^2}{|a^2 - c^2|},$$

$$DQ = \delta$$

$$= \frac{abc + dc^2}{|a^2 - c^2|}.$$

3. Length of Radii of the Circle Semi-excircle

Length of radii of tangents circle quadrilateral

A convex quadrilateral of $ABCD$ has four excircle. Each of the excircle of convex quadrilateral can be calculated the length of radii. Since we have been able to determine the length of the side of BP, CP, CQ and DQ, the length of radii of semi-excircle offensive side of BC can be used the radii of incircle on ΔBCP, that is,

$$R_b = (s_2 - \alpha)\tan\frac{1}{2}\angle BCP$$

$$= (s_2 - \beta)\tan\frac{1}{2}\angle CBP$$

$$= (s_2 - b)\tan\frac{1}{2}\angle BPC,$$

where $s_2 = 1/2(b + \alpha + \beta)$.

The value of R_b is calculated in full by using the value of a, b, c and d. For example, if substituted α and β to S_2, are obtained

$$s_2 = \frac{1}{2}(b + \alpha + \beta)$$

$$= \frac{1}{2}\left(b + \frac{bcd}{d^2 - b^2} + \frac{abd + cb^2}{d^2 - b^2}\right)$$

$$= \frac{1}{2}\left(\frac{d^2b + cb^2 + b^2a + bcd + abd - b^3}{d^2 - b^2}\right). \qquad (2.1)$$

The substitution of equation (2.1) and the value of α to the value of R_b above, it will be obtained

$$R_b = (s_2 - \alpha)\tan\frac{1}{2}\angle BCP$$

$$= \left(\frac{1}{2}\left(\frac{d^2b + b^2c + ab^2 + bcd + abd - b^3}{d^2 - b^2}\right) - \frac{bcd + ab^2}{d^2 - b^2}\right)\tan\frac{1}{2}\angle BCP,$$

$$= \frac{1}{2}\left(\frac{bd^2 + b^2c - ab^2 - b^3 + abd - bcd}{d^2 - b^2}\right)\tan\frac{1}{2}\angle BCP.$$

In another form and with a similar reduction but by using the $\angle CBP$ will be obtained

$$R_b = (s_2 - \beta)\tan\frac{1}{2}\angle CBP$$

$$= \left(\frac{1}{2}\left(\frac{d^2b + b^2c + ab^2 + bcd + abd - b^3}{d^2 - b^2}\right) - \frac{abd + cb^2}{d^2 - b^2}\right)\tan\frac{1}{2}\angle CBP$$

$$= \frac{1}{2}\left(\frac{bd^2 + b^2a - cb^2 - b^3 - abd + bcd}{d^2 - b^2}\right)\tan\frac{1}{2}\angle CBP.$$

But if using and $\angle BPC$, then the result will be slightly different, that is,

$$R_b = (s_2 - \gamma)\tan\frac{1}{2}\angle BCP$$

$$= \left(\frac{1}{2}\left(\frac{d^2b + b^2c + ab^2 + bcd + abd - b^3}{d^2 - b^2}\right) - b\right)\tan\frac{1}{2}\angle BPC$$

$$= \frac{1}{2}\left(\frac{b^2c + ab^2 - bd^2 + bcd + abd - 3b^3}{d^2 - b^2}\right)\tan\frac{1}{2}\angle BPC.$$

Furthermore, to calculate the length of R_c, can be used incircle of ΔCDQ, by using the formula

$$R_c = (s_3 - \gamma)\tan\frac{1}{2}\angle CDQ$$

$$= (s_3 - \delta)\tan\frac{1}{2}\angle DCQ$$

$$= (s_3 - c)\tan\frac{1}{2}\angle CQD,$$

where $s_3 = 1/2(c + \gamma + \delta)$.

By using the value of a, b, c and d. For example, if substituted the value γ and δ to s_3, will be obtained

$$s_3 = \frac{1}{2}\left(\frac{acd + bc^2}{a^2 - c^2} + \frac{abc + dc^2}{a^2 - c^2} + c\right)$$

$$= \frac{1}{2}\left(\frac{dc^2 + bc^2 + ca^2 + acd + abc - c^3}{a^2 - c^2}\right).$$

Thus obtained

$$R_c = (s_3 - \gamma)\tan\frac{1}{2}\angle CDQ$$

$$= \left(\frac{1}{2}\left(\frac{dc^2 + bc^2 + ca^2 + acd + abc - c^3}{a^2 - c^2}\right) - \frac{abc + dc^2}{a^2 - c^2}\right)\tan\frac{1}{2}\angle CDQ$$

$$= \frac{1}{2}\left(\frac{dc^2 - bc^2 + ca^2 + acd - abc - c^3}{a^2 - c^2}\right)\tan\frac{1}{2}\angle CDQ.$$

If using $\angle DCQ$, will be obtained

$$R_c = (s_3 - \delta)\tan\frac{1}{2}\angle DCQ$$

$$= \left(\frac{1}{2}\left(\frac{dc^2 + bc^2 + ca^2 + acd + abc - c^3}{a^2 - c^2}\right) - \frac{abc + dc^2}{a^2 - c^2}\right)\tan\frac{1}{2}\angle DCQ$$

$$= \frac{1}{2}\left(\frac{bc^2 + ca^2 - dc^2 - c^3 + abd - abc}{a^2 - c^2}\right)\tan\frac{1}{2}\angle DCQ.$$

Meanwhile, if using $\angle CQD$, will obtain slightly different results, namely

$$R_c = (s_3 - c)\tan\frac{1}{2}\angle CQD$$

$$= \left(\frac{1}{2}\left(\frac{dc^2 + bc^2 + ca^2 + acd + abc - c^3}{a^2 - c^2}\right) - c\right)\tan\frac{1}{2}\angle CQD$$

$$= \frac{1}{2}\left(\frac{dc^2 + bc^2 - ca^2 + acd + abc - 3c^3}{a^2 - c^2}\right)\tan\frac{1}{2}\angle CQD.$$

While to calculating R_d and R_b, each of which is the same as counting excircle of ΔAPD and ΔABQ, since

$$s_4 = \frac{1}{2}(a + \alpha + \beta + c + d)$$

and

$$s_1 = \frac{1}{2}(b + \gamma + \delta + d + a)$$

so

$$s_4 = \frac{1}{2}((a + \alpha) + (\beta + c) + d)$$

$$= \frac{1}{2}\left(\left(a + \frac{bcd + ab^2}{d^2 - b^2}\right) + \left(\frac{abd + cb^2}{d^2 - b^2} + c\right) + d\right)$$

$$= \frac{1}{2}\left(\frac{ad^2 + d^2c - db^2 + d^3 + bcd + abd}{d^2 - b^2}\right)$$

and

$$s_1 = \frac{1}{2}(a + (b + \gamma) + (\delta + d))$$

$$= \frac{1}{2}\left(a + \left(b + \frac{acd + bc^2}{a^2 - c^2}\right) + \left(\frac{abc + dc^2}{a^2 - c^2} + d\right)\right)$$

$$= \frac{1}{2}\left(\frac{da^2 + ba^2 - c^2a + a^3 + acd + abc}{a^2 - c^2}\right).$$

Thus obtained

$$R_d = \frac{1}{2}\left(\frac{d^2a + d^2c - db^2 + d^3 + bcd + abd}{d^2 - b^2}\right)\tan\frac{1}{2}\angle APD$$

and

$$R_a = \frac{1}{2}\left(\frac{da^2 + bc^2 - c^2a + a^3 + acd + abc}{a^2 - c^2}\right)\tan\frac{1}{2}\angle BQA.$$

As said above, the value of R_a, R_b, R_c and R_d is for case side lengths of $a > c$ and the long side of $d > b$ so if otherwise applicable, then the denominator from the equation above for $a^2 - c^2$ changed into $c^2 - a^2$ and $d^2 - b^2$ changed into $b^2 - d^2$. So generally will be obtained

$$R_a = \frac{1}{2}\left(\frac{da^2 + bc^2 - c^2a + a^3 + acd + abc}{|a^2 - c^2|}\right)\tan\frac{1}{2}\angle BQA,$$

$$R_b = \frac{1}{2}\left(\frac{bd^2 + b^2c - ab^2 - b^3 + abd - bcd}{|d^2 - b^2|}\right)\tan\frac{1}{2}\angle BCP,$$

$$R_b = \frac{1}{2}\left(\frac{bd^2 + b^2a - cb^2 - b^3 - abd + bcd}{|d^2 - b^2|}\right)\tan\frac{1}{2}\angle CBP,$$

$$R_b = \frac{1}{2}\left(\frac{b^2c + ab^2 - bd^2 + bcd + abd - 3b^3}{|d^2 - b^2|}\right)\tan\frac{1}{2}\angle BPC,$$

$$R_c = \frac{1}{2}\left(\frac{dc^2 - bc^2 + ca^2 + acd - abc - c^3}{|a^2 - c^2|}\right)\tan\frac{1}{2}\angle DCQ,$$

$$R_c = \frac{1}{2}\left(\frac{bc^2 + ca^2 - dc^2 - c^3 + abd - abc}{|a^2 - c^2|}\right)\tan\frac{1}{2}\angle CDQ,$$

$$R_c = \frac{1}{2}\left(\frac{dc^2 + bc^2 - ca^2 + acd + abc - 3c^3}{|a^2 - c^2|}\right)\tan\frac{1}{2}\angle CQD,$$

$$R_d = \frac{1}{2}\left(\frac{d^2a + d^2c - db^2 + d^3 + bcd + abd}{|d^2 - b^2|}\right)\tan\frac{1}{2}\angle APD.$$

On the above calculations, there are respectively 3 formulas to calculate the value of R_b and R_b, viewed from a different angle. While the formula to calculate the value of R_a and R_b each only one formula. This is because the formula for calculating R_a and R_b used excircle formula, which the excirle formula only unique. While the formula to calculate R_d and R_c derived from the incircle formula, which incircle formula basically can be derived from three concepts of that angle.

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