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Optimal Control Strategy for Alcoholism Model with Two Infected Compartments

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Abstract : *This article discussed mathematical model of alcoholism with optimal control.Model is designed using system of differential equations based on Susceptible, Infectible and Resistant (SIR) model. Infected individuals is divided into two compartments, admitted and non-admitted to alcoholism. Optimal control is used to prevent interaction between susceptible individual and infected individuals. Stability analysis is done locally using Routh-Hurwitz criteria. It can be shown that optimal control determines stability of the system. In the end of article, numerical simulation is given to illustrate uncontrolled and controlled system. The results show optimal control succeed to reduce infected individuals. Controlled system has higher susceptible individuals and has lower infected individuals than uncontrolled system.*

Keywords *- Alcoholism model, local stability, optimal control, Routh-Hurwitz criteria, SIR model*

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I. Introduction

Liquor distribution is restricted in Indonesia. The state regulation has set some restriction regarding the age of consumers, the dispersal area and the alcohol content. The regulation is made because the harmful effects of alcohol. Unfortunately, there are many liquor distributions illegally containing hazardous materials. On April, 2018, it was reported that 51 people was died as a result of consuming liquor [1]. They consume liquor which is self formulated. They combine liquor with hazardous materials like insecticide and tonic.

Some research are done about alcoholism. Bhunu [2] analyze alcoholism model which is linear model using SEIR scheme. In this model, infected individuals is divided into moderate drinker and alcoholic. Mulone and Straughan [3] analyze binge model based on SIR model. They propose two model, which are standard model with single compartment of infected individuals and modified model with two compartments of infected individuals. Both of them analyze dynamical behavior of alcoholism model without treatment or vaccination. Wang et al [4] propose and analyze nonlinear alcoholism model. Their model is developed using SIR model. Optimal control is used to control infected individuals using treatment strategy. It is worth noting that the work presented here differs from [4] in that this model combine model in [3] using optimal control.

In this paper, we consider nonlinear model of alcoholism using Susceptible, Infected and Resistant (SIR) model combining with optimal control. Optimal control is used to prevent interaction between susceptible individuals and infected individuals. Our aim is to construct an optimal control to minimize infected individuals. For this, we start with the model proposed by [3] which is a dynamical system then we modify it to be more realistic, using optimal control to restrain the successful transmission rate of infection in consequence of interaction between infected and susceptible individuals.

This paper is organized as follow. In section 2, we present a mathematical model describing influence of alcoholism with control optimal term. The stability analysis is presented in section 3. In section 4, we design control term using optimal control. Numerical simulation with certain value parameter to show behavior uncontrolled and controlled system is shown in section 5. Finally, the conclusion is summarized in section 6.

II. The Model Formulation

In this section, mathematical model of alcoholic is obtained based on [3]. Our population is divided into $S(t)$, $A_1(t)$, $A_2(t)$ and $R(t)$. $S(t)$ is number of susceptible individuals, $A_1(t)$ is number of infected individuals which hide to drink liquor, $A_2(t)$ is number of infected individuals which admit to drink liquor and $R(t)$ is number of recovered individuals. It is assumed that recovery population will not drink liquor anymore. In this model, optimal control is used to prevent susceptible individuals turn into infected individuals. Susceptible individuals will turn into infected if interaction between susceptible and infected population is occur. Control $u(t)$ is given between susceptible individuals and both of infected individuals to reduce rate of infection. The corresponding scheme of the systems is illustrated in Figure 1.

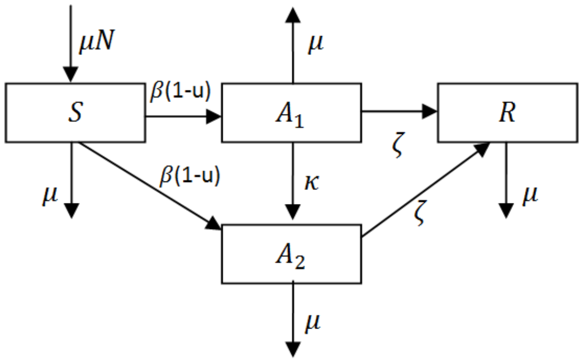


Fig 1: Transfer diagram for the model with control between susceptible and infected individuals

Based on Figure 1, optimal control of alcoholic model is governed by the following system of differential equations

$$
\begin{aligned}
\frac{dS}{dt} &= \mu N - \frac{\beta(1-u)S(A_1+A_2)}{N} - \mu S \\
\frac{dA_1}{dt} &= \frac{\beta(1-u)SA_1}{N} - \mu A_1 - \kappa A_1 - \zeta A_1 \\
\frac{dA_2}{dt} &= \frac{\beta(1-u)SA_2}{N} - \mu A_2 + \kappa A_1 - \zeta A_2 \\
\frac{dR}{dt} &= \zeta(A_1+A_2) - \mu R
\end{aligned}
\qquad (1)
$$

where all variables and parameters are positive and described in Table 1.

Table. 1 Variables and Parameters

Parameter	Definition	Dimension
$S(t)$	Number of susceptible individuals	Human
$A_1(t)$	Number of admitted infected individuals	Human
$A_2(t)$	Number of non-admitted infected individuals	Human
$R(t)$	Number of recovered individuals	Human
μ	Human recruitment or natural death of individuals	1/time
β	Successful transmission rate for infected individuals	1/time
κ	Transition rate of individuals from non-admitted infected to admitted infected individuals	1/time
ζ	Transition rate of individuals infected to recovered individuals	1/time

$N(t)$ denotes total populations, that is $N = S + A_1 + A_2 + R$. From Equation (1), we achieve $\frac{dN}{dt} = 0$, which mean that total number of populations is constant all the time. For convenience, we introduce new variables

$$s = \frac{S}{N}, a_1 = \frac{A_1}{N}, a_2 = \frac{A_2}{N}, r = \frac{R}{N}$$

and $s + a_1 + a_2 + r = 1$ for transforming Equation (1) into non dimensional model as state as follow

$$\begin{aligned}
\frac{ds}{dt} &= \mu - \beta(1-u)\,s\,(a_1+a_2) - \mu s \\
\frac{da_1}{dt} &= \beta(1-u)\,sa_1 - (\mu+\kappa+\zeta)a_1 \\
\frac{da_2}{dt} &= \beta(1-u)\,sa_2 - (\mu+\zeta)a_2 + \kappa a_1 \\
\frac{dr}{dt} &= \zeta(a_1+a_2) - \mu r
\end{aligned} \tag{2}$$

Note that $r(t)$ in (2) have no an effect to others equations. In next analysis, we consider to analysis system consist of $s(t), a_1(t), a_2(t)$.

$$\begin{aligned}
\frac{ds}{dt} &= \mu - \beta(1-u)\,s\,(a_1+a_2) - \mu s \\
\frac{da_1}{dt} &= \beta(1-u)\,sa_1 - (\mu+\kappa+\zeta)a_1 \\
\frac{da_2}{dt} &= \beta(1-u)\,sa_2 - (\mu+\zeta)a_2 + \kappa a_1
\end{aligned} \tag{3}$$

III. Equilibrium Points and Stability Analysis

Stability of Equation (3) is analyzed linearly using Routh-Hurwitz criteria. We transform nonlinear system of differential equation in (3) into linear system using Taylor series about equilibrium points. In this paper, we consider only two kind of equilibrium points, those are infected free and endemic equilibrium point. It will be shown that the controls have an effect on the stability of the model.

It is clear that infected free equilibrium point $(1,0,0)$ hold Equation (3). This is obtained from Equation (3) and setting leftside with 0 and assuming $a_1 = a_2 = 0$. Endemic equilibrium point is given by $\left(s^*, a_1^*, a_2^*\right)$ with

$$s^* = \frac{\mu+\zeta}{\beta(1-u)}$$

$$a_1^* = 0$$

$$a_2^* = \frac{\mu}{\beta}\,\frac{(u-1)\beta+\zeta+\mu}{(\zeta+\mu)(u-1)}$$

Proposition 1. *Disease free equilibrium point is locally asymptotically stable if control u(t)* $u > 1 - \dfrac{\mu+\zeta}{\beta}$.

Proof. Jacobian matrix from Equation (3) is

$$A = \begin{pmatrix} -H(a_1+a_2)-\mu & -Hs & -Hs \\ Ha_1 & Hs-(\mu+\kappa+\zeta) & 0 \\ Ha_2 & \dfrac{\kappa}{H} & Hs-(\mu+\zeta) \end{pmatrix} \tag{4}$$

with $H = \beta(1-u(t))$. Substitute infected free equilibrium point $E_0 = (1,0,0)$ to Equation (4) will produce

$$A_0 = \begin{pmatrix} -\mu & -\beta(1-u) & -\beta(1-u) \\ 0 & \beta(1-u)-(\mu+\kappa+\zeta) & 0 \\ 0 & \dfrac{\kappa S}{\beta(1-u)} & \beta(1-u)-(\mu+\zeta) \end{pmatrix} \tag{5}$$

Characteristic polynomial of Equation (5) is obtained from $\det\left(A_0-\lambda I\right)=0$ where I is a square identity matrix of order 3. The equation thus become

$$(\lambda+\mu)\left(\lambda+\beta(u-1)+\mu+\zeta\right)\left(\lambda+\beta(u-1)+\kappa+\mu+\zeta\right)=0 \qquad (6)$$

The roots of characteristic polynomial in (5) are the eigen values of matrix A_0 in (5). Solutions of Equation (6) given by

$$\lambda_0=\begin{pmatrix}-\mu\\ (1-u)\beta-(\mu+\zeta)\\ (1-u)\beta-(\mu+\zeta+\kappa)\end{pmatrix} \qquad (7)$$

Second and third eigen values of matrix A_0 in (7) show that control $u(t)$ determines the sign of eigen values λ_0 in (7). It means that control $u(t)$ can change behavior of the system near equilibrium point E_0. Based on parameter assumption, value of the first eigen value in λ_0 is negative while the second and third eigen value can not be determined yet. If we want the system (3) locally asymptotically stable about E_0 then the sign of the second and third of λ_0 must be negative. It results

$(1-u)\beta<\mu+\zeta<\mu+\zeta+\kappa$

So, necessary condition for locally asymptotically stable system is $u\geq 1-\dfrac{\mu+\zeta}{\beta}$.

Corollary 2. *System of differential equations in (3) is locally asymptotically stable about infected free equilibrium point if system is fully controlled.*

Corollary 2 is obtained if full control $u(t)=1$ is given to the system at any time so that eigen value $\lambda_{0,2}=-(\mu+\zeta)$ and $\lambda_{0,3}=-(\mu+\zeta+\kappa)$ always negative using parameter assumption. Stability of the system in (3) without control at any time is shown in [3].

Theorem 3. *Endemic equilibrium point is locally asymptotically stable if control* $u<1-\dfrac{\mu+\zeta}{\beta}$.

Bukti. The corresponding Jacobian matrix about E_1 is as follows

$$A_1=\begin{pmatrix}\dfrac{\mu\beta(u-1)}{\mu+\zeta} & -\mu-\zeta & -\mu-\zeta\\ 0 & -\kappa & 0\\ \dfrac{\mu\beta(u-1)}{\mu+\zeta}-\mu & \kappa & 0\end{pmatrix} \qquad (8)$$

Characteristic polynomial of matrix A_1 in (8) is

$$\lambda^3-\frac{\mu\beta(u-1)-(\mu+\zeta)\kappa}{\mu+\zeta}\lambda^2-\frac{\mu\left(\beta(u-1)(\kappa+\mu+\zeta)+(\mu+\zeta)^2\right)}{\mu+\zeta}\lambda-\kappa\mu\left((u-1)\beta+\mu+\zeta\right)=0 \qquad (9)$$

The solution of Equation (9) which is the eigen value of the matrix A_1 is

$$\lambda_1=\begin{pmatrix}-\kappa\\ -\dfrac{\psi-\sqrt{\theta}}{\mu+\zeta}\\ -\dfrac{\psi+\sqrt{\theta}}{\mu+\zeta}\end{pmatrix} \qquad (10)$$

with $\psi = \beta\mu(1-u)$ and θ given by

$$\theta = \mu\beta(u-1)\left(\mu\beta(u-1)+4(\zeta+\mu)^2\right)+4\mu(\zeta+\mu)^3 . \qquad (11)$$

Based on parameter assumption, clearly the eigen values in (10) has negative value except second eigen value. Thus, define $\tau = \psi^2 - \theta$. Substituting value of θ in Equation (11) to τ will result

$\tau = -4\mu(\zeta+\mu)^2(\beta u+\zeta-\beta+\mu)$

If desired system to be locally asymptotically stable in E_1 then $\tau > 0$ which is fulfilled by $\beta(u-1)+\zeta+\mu<0$ or $u < 1 - \dfrac{\mu+\zeta}{\beta}$.

IV. Design optimum control

In this section we use optimal control theory to analyze behavior of the model (3). Our goal is to increase susceptible individual and reduce the cost of prevention process. The first step is defining objective function to be minimized. Mathematically, the problem is to minimize the objective functional

$$J(s,a_1,a_2,u) = \int_0^{t_f} a_1(t)+a_2(t)+\phi u^2(t)\,dt \qquad (12)$$

where the parameter ϕ is positive constant denotes weighting factor on cost. Value of ϕ determine by control designer using trial and error to adjust with desire output. The goal to be achieved in this procedure is to minimize cost functional $J(s,a_1,a_2,u)$ in Equation (12) subject to system of differential equations in (3).

Theorem 4. *There exists an optimal control $u^*(t)$ and corresponding solution s, a_1, a_2 that minimize $J(s,a_1,a_2,u)$. Furthermore, the optimal control $u^*(t)$ is*

$$u^*(t) = \min\left(1, \max\left(0, u(t)\right)\right) \qquad (13)$$

with $u(t)$ is given by

$$u(t) = \frac{s}{2\phi}\left(\lambda_3 a_2 + \lambda_2 a_1 - \lambda_1\beta\left(a_1+a_2\right)\right) \qquad (14)$$

Bukti: The existence of optimal control can be proved using the result form [5]. Next, we define Hamiltonian equation

$$H = a_1(t)+a_2(t)+\phi u^2(t)+\sum_{i=1}^{3}\lambda_i f_i \qquad (15)$$

where λ_i are Lagrange multiplier and f_i are function on right side in Equation (3). Substitute each function in right side in Equation (3) to (15) yield

$$\begin{aligned} H &= a_1(t)+a_2(t)+\phi u^2(t)+\lambda_1\left(\mu-\beta(1-u)s(a_1+a_2)-\mu s\right) \\ &+\lambda_2\left(\beta(1-u)sa_1-(\mu+\kappa+\zeta)a_1\right)+\lambda_3\left(\beta(1-u)sa_2-(\mu+\zeta)a_2+\kappa a_1\right) \end{aligned} \qquad (16)$$

Optimal control $u(t)$ that minimize H in Equation (16) can be achieved with partial derivative H respect to $u(t)$. Differentiate H respect to $u(t)$ yield

$$\frac{\partial H}{\partial u} = 2\phi u+\lambda_1\beta s(a_1+a_2)-\lambda_2 sa_1-\lambda_3 sa_2 = 0 \qquad (17)$$

Solving Equation (17), value of $u(t)$ that minimize H in Equation (16) is

$$u(t) = \frac{s}{2\phi}\left(\lambda_3 a_2 + \lambda_2 a_1 - \lambda_1\beta\left(a_1+a_2\right)\right) \qquad (18)$$

$u(t)$ in Equation (18) always well defined in interval $t\in[0,\infty)$ because $u(t)$ is an polynomial. Moreover, $u(t)$ is bounded below by zero (0) when it is uncontrolled condition then we defined control $u^*(t)$

$$u^{*}(t) = \max\{0, u(t)\} \qquad (19)$$

We assume control $u(t)$ is bounded above and based on Fig. 1 maximum value of $u(t)$ is $u(t) = 1$. Thus, control $u^{*}(t)$ in Equation (19) can be expressed by

$$u^{*}(t) = \min\{1, \max\{0, u(t)\}\} \qquad (20)$$

Values of λ_i Equation (18) is obtained from costate equation obtained by partial derivative function H in equation (16) respect to each variables s, a_1 and a_2. Differentiate H respect to s yield λ_1

$$-\lambda_1' = \beta(u-1)\left(\lambda_1(a_1 + a_2 + \mu) + \lambda_2 a_1 + \lambda_3 a_2\right) \qquad (21)$$

Value of λ_2 is obtained by $\frac{\partial H}{\partial a_1} = -\lambda_2'$

$$-\lambda_2' = 1 + \lambda_1\left(-\beta(1-u)s\right) + \lambda_2\left(\beta(1-u)s - (\mu+\kappa+\zeta)\right) + \lambda_3\kappa \qquad (22)$$

Differentiate H respect to a_2 yield λ_3

$$-\lambda_3' = 1 - \lambda_1\beta(1-u)s + \lambda_3\left(\beta(1-u)s - \mu - \zeta\right) \quad (23)$$

Equation (21), (22) and (23) are system of differential equations with boundary conditions $\lambda_1(t_f) = 0$, $\lambda_2(t_f) = 0$ and $\lambda_3(t_f) = 0$

System of differential equations (3) with optimal control (13) and (14) is solved using following step

1. First step, assuming $u(t) = 0$ which mean that system is not controlled.
2. Solve system of differential equations in Equation (3) numerically with given initial conditions.
3. Solve system of differential equations in Equation in Equation (21), (22) and (23) numerically with boundary condition.
4. Update value of control $u(t)$ using Equation (13) and (14).
5. Do step (2) to step (4) until stopping criterion are met.

V. Numerical Simulation

This section illustrate effect of control (13) to system differential equation (3). The value of parameters for simulation are given in Table 2 and initial condition are $s_0 = 0.8, a_1 = 0.1, a_2 = 0.05, r = 0.05$. First, we do numerical simulation with certain value $\phi = 100$. Thus, we do numerical simulation with different values of ϕ to show effect the weighting. Values of ϕ for comparison are $\phi = \{0.01, 40, 100\}$

Table. 2 Value of parameters for numerical simulation

Parameter	Values
μ	0.2
β	0.5
κ	0.051
ζ	0.1
ϕ	100

Based on parameter Table 2, stability requirement for infected free equilibrium point is $u > 0.4$ and endemic equilibrium point is $u < 0.4$. We use Matlab with ODE45 to solve system of differential equations numerically with relative tolerance $1e^{-4}$ and iterate the procedure for 10 times.

First, we compare controlled and uncontrolled system that met the condition in Theorem (1) and (2) with constant $u = \{0, 0.25, 0.65\}$. These constant value of $u = \{0, 0.25, 0.65\}$ represents uncontrolled system, controlled system that met stable endemic condition and controlled system that met stable infected free condition. Solution $s(t), a_1(t), a_2(t)$ are shown in Figure 2 and 3. Figure 2 show $s(t)$ is plotted against time with different and constant $u(t)$. With increase time, $s(t)$ in uncontrolled system decrease. When system is controlled by $u = 0.25 < 0.4$, $s(t)$ is higher than uncontrolled system but lower than $s(t)$ with $u = 0.65 > 0.4$. It show that the greater control, the higher $s(t)$. Figure 3 show plot $a_1(t), a_2(t)$ against time. Controlled system has lower infected individuals than uncontrolled system. Figure 3.b show that control $u = 0.25$ is not enough to make infected population vanish.

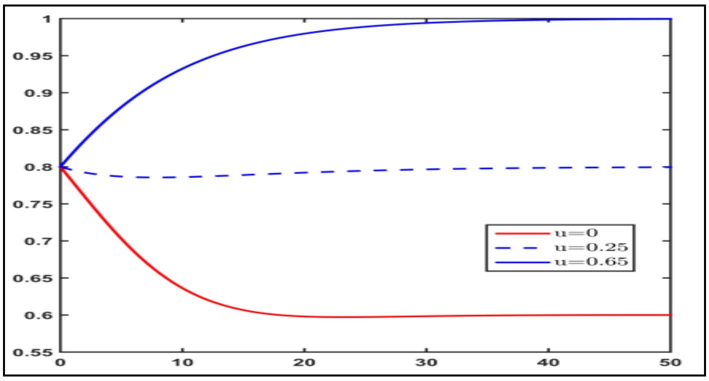
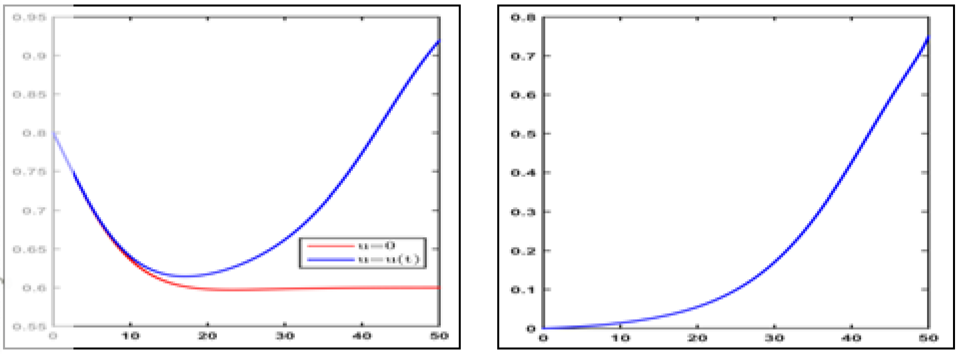


Fig 2: Comparison $s(t)$ without control and controlled with different and constant control $u = \{0, 0.25, 0.65\}$



(a) $s(t)$ (b) time varying $u(t)$

Fig 3: Comparison $a_1(t), a_2(t)$ without control and with different and constant control $u = \{0, 0.25, 0.65\}$

Next, effect of different controls on the stability susceptible populations $s(t)$ is simulated. To show behavior population near endemic equilibrium point, we choose $u = \{0.25, 0.35\}$ and about infected free equilibrium point we use $u = \{0.65, 0.85\}$. Figure 4 show that the greater control $u(t)$ given, the higher the susceptible populations. Unfortunately, higher control will produce higher cost. Table 3 shows comparison cost $J(s, a_1, a_2, u)$ as the result of controlling system with different value of control $u(t)$. From these result, we can state that if we want to make system converge to desire path faster, we can do that by setting control in higher value but we have to pay more expensive for that work.

Table. 3 Comparison value of $J(s, a_1, a_2, u)$ using different control

u	J ($1e^{-5}$)
0.25	1.2569
0.35	1.7543
0.65	3.2510
0.85	4.2506

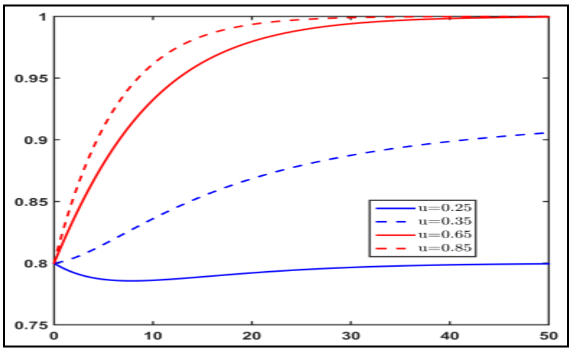
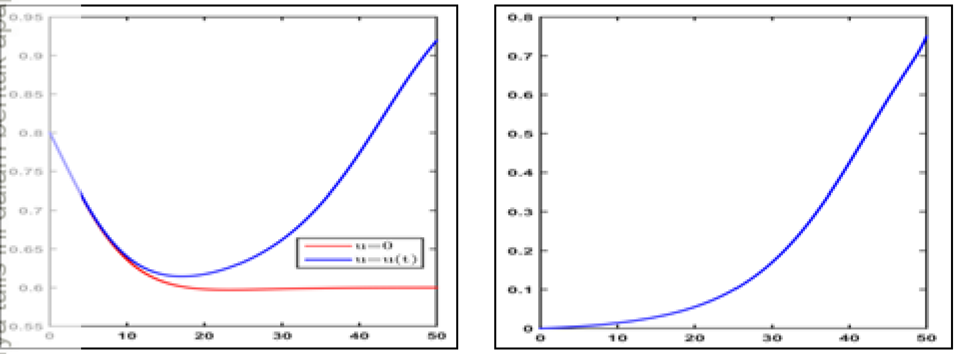


Fig 4: Comparison population $s(t)$ with different and constant control $u=\{0.25, 0.35, 0.65, 0.85\}$

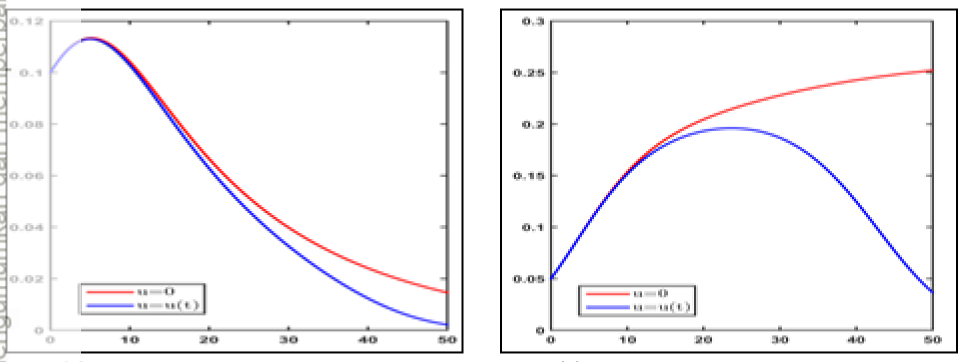
Next we simulate behavior of the system (3) based on algorithm proposed in this article. We also show that control $u(t)$ produce lower value of $J(s, a_1, a_2)$ than previous simulations. The results are shown in Figure 5 and 6. Figure 5.a show controlled system has higher $s(t)$ than uncontrolled systems. Figure 5.b show value of $u(t)$.



(a) $s(t)$ (b) time varying $u(t)$

Fig 5: (a) Comparison behavior uncontrolled and controlled $s(t)$. (b) Value $u(t)$ in $t \in [0, 50]$

Figure 6 show behavior of infected populations. Figure 6.a show non-admitted alcoholism model $a_1(t)$ increase at the beginning of time then decrease toward zero. Difference between uncontrolled and controlled system is not significance but it is clear controlled system has lower value than uncontrolled system. Figure 6.b show uncontrolled admitted alcoholic model always increase at any time. Population $a_2(t)$ in controlled system increase at the beginning of time but decrease and toward zero when $t = 50$. Controlled system produce 1.029e5 which is smallest than all of the cost in Table 3. From these results we can state that optimal control succeed reduce number of infected individuals.



(a) $a_1(t)$ (b) $a_2(t)$

Fig 6: Comparison behavior $a_1(t), a_2(t)$ without control and controlled by value of $u(t)$ shown is Figure 5b.

Cost $J(s,a_1,a_2,u)$ is influenced by weighting factor ϕ. The last simulation use different $\phi = \{0.01, 40, 100\}$ to analyze change value of cost $J(s,a_1,a_2,u)$ respect to ϕ. The results are shown in Figure 7, 8 and cost $J(s,a_1,a_2,u)$ of each ϕ is shown in Table 5.

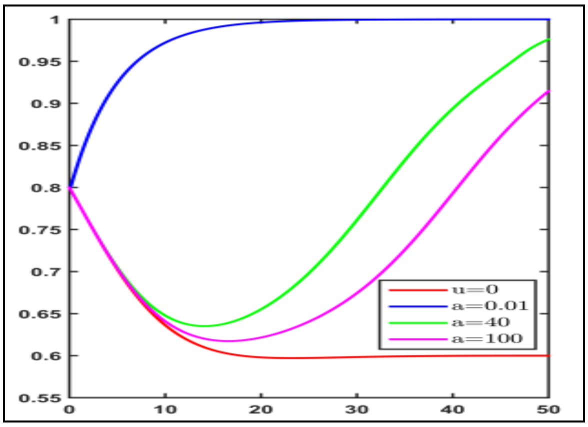
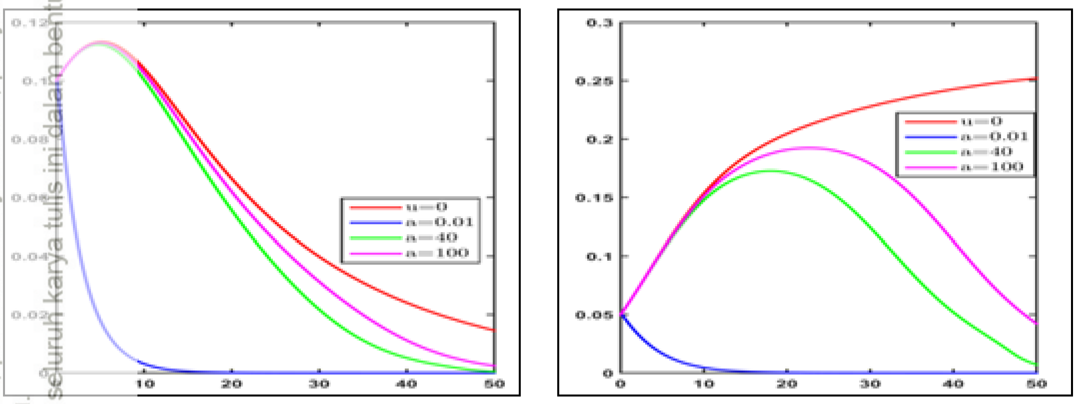


Fig 7: Behavior of $s(t)$ with control and various value of ϕ



(a) $a_1(t)$ (b) $a_2(t)$

Fig 8: Behavior of infected individual $a_1(t), a_2(t)$ with control and various value of ϕ

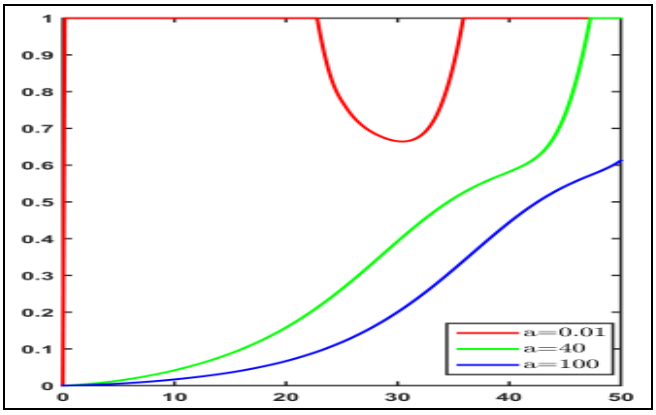


Fig. 9 Various value of control $u(t)$ with different value of ϕ

Table. 5 Comparison $J(s,a_1,a_2,u)$ with various ϕ

ϕ	J $(1e5)$
0.01	0.0009
40	0.5581
100	1.0290

VI. Conclusion

Optimal control for alcoholic model with two compartments is discussed in this paper. Control optimum is used to prevent interaction between susceptible individual and infected individuals. From stability analysis, it is shown that control optimum succeed to decrease infected individuals and increase susceptible individuals. Various value of ϕ is also simulated and the results show that smaller value of weighting is given, smaller value of cost $J\left(s, a_1, a_2, u\right)$.

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