Root Mean Square Newton's Method

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Abstract

In this paper we discuss a new modification of Newton's method based on Root Mean Square rule for solving nonlinear equations. We show that the convergence properties of the proposes method is of order three. We verify the theoretical results on relevant numerical problems and compare the behavior of the propose method with some mean based Newton's method.

Keywords: Arithmetic Mean, Geometry Mean, Harmonic Mean, Heronian Mean, Root Mean Square.

Introduction

Solving non-linear equations is one of classical problem in numerical analysis. In this paper, we consider an iterative method to find the solution(s) of the nonlinear equation

$$f(x)=0, \quad f:D\subseteq R\to R. \tag{1}$$

Newton methods for a single non-linear equation is written as

$$x_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)} \tag{2}$$

where x_n is the nth approximation of the root α. This is an important and basic method which converges quadratically.

Recently, some of new modified Newton's method (NM) with cubic convergence have been developed in [3] [5] [6] [10], by considering different quadrating formulae for the computation of the integral arising from Newton's theorem

$$f(x)=f(x_n)+\int_{x_n}^{x} f'(s)\,ds \tag{3}$$

Weerakoon and Fernando [10] approximate the indefinite integral involved in (3) by the trapezoidal rule,

$$\int_{x_n}^{x} f'(s)ds=\left(\frac{f'(x_n)+f'(x)}{2}\right)(x-x_n) \tag{4}$$

and then by some algebra they end up with the following method

$$x_{n+1}=x_n-\frac{2f(x_n)}{f'(x_n)+f'(x_{n+1}^{*})}, \qquad n=0,1,2,\cdots, \tag{5}$$

$$x^{*}_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

(6)

They prove that the method is a third order convergence. The way they choose x^{*}_{n+1} as in (7) was introduced for the first time by Wall [11].

Ozban [8] replaced arithmetic mean with harmonic mean in (5), that is

$$\int_{x_n}^{x} f'(s)ds = \left(\frac{2f'(x_n)f'(x)}{f'(x_n)+f'(x)}\right)(x - x_n)$$

He obtain Harmonic mean Newton's method (HN)

$$x_{n+1} = x_n - \frac{f(x_n)(f'(x_n)+f'(x^{*}_{n+1}))}{2f'(x_n)f'(x^{*}_{n+1})}, \qquad n = 0,1,2,\cdots,$$

(7)

$$x^{*}_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

(8)

Similarly Lukic and Ralevic [9] use geometric mean instead of arithmetic mean in (4), the find a formula for Geometric mean Newton's method (GM), as follows:

$$x_{n+1} = x_n - \frac{f(x_n)}{\mathrm{sign}(f(x_0))\sqrt{f'(x_n)f'(x^{*}_{n+1})}}, \qquad n = 0,1,2,\cdots,$$

(9)

$$x^{*}_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

(10)

Imran [12] substitute heronian mean to replace arithmetic mean in (4). He obtain Heronian mean Newton's method (HeM) as follows:

$$x_{n+1} = x_n - \frac{3f(x_n)}{f'(x_n)+f'(x^{*}_{n+1})+\mathrm{sign}(f(x_0))\sqrt{f'(x_n)f'(x^{*}_{n+1})}}, \qquad n = 0,1,2,\cdots,$$

(11)

$$x^{*}_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

(12)

They show that this formula is of order three for simple root and linear for multiple roots.

In this study we suggest a modification of the iteration of Newton's method by approximating the indefinite integral using Root mean square and we derive Root mean square Newton's method. At the end we do some numerical experiments for Mean Based Newton's method.

Root Mean Square Newton Method

To derive the method, we consider the computation of the indefinite integral on a new interval of integration arising from Newton's theorem

$$f(x) = f(x_n) + \int_{x_n}^{x} f'(s)\, ds$$

(13)

Now we use the Root mean square to approximate the right integral of (13)

$$\int_{x_n}^{x} f'(s)\, ds \approx \left(\frac{\sqrt{(f'(x_n))^2 + (f'(x))^2}}{2} \right)(x - x_n) \qquad (14)$$

and looking for $f(x) = 0$, we obtain a new method

$$x_{n+1} = x_n - \frac{\sqrt{2}\, f(x_n)}{\sqrt{(f'(x_n))^2 + (f'(x_{n+1}))^2}}. \qquad (15)$$

From (15) we propose Root Mean Square Newton's method (RMS) as follows:

$$x_{n+1} = x_n - \frac{\sqrt{2}\, f(x_n)}{sign(f'(x_0))\sqrt{(f'(x_n))^2 + \left(f'(x^*_{n+1})\right)^2}}, \quad n = 1,2,\cdots \qquad (16)$$

$$x^*_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

(17)

Comparison

3.1 Analytical Comparisons

For analytical comparisons we look into the cost of function evaluation for the methods, as in Table 1. From this Table, we see that all methods need only one function evaluation. The Mean-Based Newton's Methods (MBN) need an expense in the derivative of f. From this point, we can say that all MBN are comparable in term of the functional evaluations. However if we look into the addition and multiplication costs, the GM is fewer than the other MBN.

Table 1. Comparisons of the computational costs of the Mean-Based Newton's Methods

Method	Cost of			
	Addition/subtraction	Multiplication/division	$f(x)$	$f'(x)$
NM	1	1	1	1
AM	3	3	1	2
HN	3	5	1	2
GM	2	3	1	2
HeM	4	4	1	2
RMS	3	5	1	2

3.2 Numerical Experiments

The MBN and NM are tested using some functions, which have been used in [10] and [9]. We compare the number of iterations for each method by varying some initial values. We also compute the computational order of convergence of the methods (2) for each initial value. We stop the program using the following criteria

$$|x_{n+1} - x_n| < \varepsilon |x_{n+1}|$$
$$|f(x_{n+1})| < \varepsilon$$

where $\varepsilon = 1.0e-12$. All computations are done using Matlab on Windows PC with Intel Processor at 2.4 GHz. The computational results are given in Table 2.

From Table 2, we see that the results of our proposed method (RMS) are comparable with the other MBN Methods. The HeM also has the same characteristics as other MBN Methods, that is:

1. third order of convergence for simple root,
2. does not require the computation of second or higher order derivatives,

Table 2. Comparisons of the number of iterations of the Mean-Based Newton's Methods

$f(x)$	x_0	Number of iterations					
		NM	AM	HM	GM	HeM	RMS
x^3+4x^2-10	0.5	6	4	3	4	4	5
	1.0	5	3	3	3	3	4
	2.0	5	3	3	3	3	4
$\sin^2 x - x^2 + 1$	3.0	6	3	3	3	3	5
	1.0	5	4	3	3	3	5
$2x\exp(-20)+1$	0.1	7	5	4	5	5	6
$-2\exp(-20x)$	0.0	5	3	3	3	3	4
$(x-1.0)^3-1$	2.5	5	4	3	3	4	4
	4.0	7	5	4	4	5	5
	1.5	7	5	4	4	4	5

NM - Newton's Method GM - Geometric Mean
Newton's Method
AM - Arithmetic Mean Newton's Method HeM - Heronian
Mean Newton's Method
HM - Harmonic Mean Newton's Method RMS – Root Mean
Square Newton's Method

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