NEWTON ITERATION BASED ON HERONIAN MEAN

M. Imran,
Laboratorium Matematika Terapan, Jurusan Matematika
Fakultas Matematika dan Ilmu Pengetahuan Alam, Universiatas Riau, Pekanbaru Indonesia
e-mail: mimran@unri.ac.id

Abstract. *We discuss modified Newton's method using heronian mean instead of arithmetic mean. We show that this modification is of order three for a simple root. Comparisons among mean based Newton's method are given by considering some numerical examples.*
Keywords: *Arithmetic mean, Geometry means, Harmonic mean, Heronian Mean, Modified Newton's Method*

1. Introduction

One of classical problem in numerical analysis is to find the solution(s) of the nonlinear equation

$$f(x) = 0, \quad f : D \subseteq R \to R. \tag{1}$$

Recently, due to the development of computer software and hardware many iterative method have been developed to approximate a solution to a nonlinear equation $f(x) = 0$: The developments were done by modifying existing methods, such as [3] [5] [6] [10], or by introducing a new method which have the same characteristics with the old methods, such as [1]. The aim of all developments is to find a method which shall convergent faster than the old ones, and is also reliable.

Weerakoon & Fernando [10] derived a modified Newton's method using arithmetic mean. His idea follows by Ozban [8] and Lukic & Ralevic [9] by introducing Harmonic and Geometry mean respectively to derive Harmonic mean Newton's method and Geometry mean Newton's method.

In this study we suggest a modification of the iteration of Newton's method by approximating the indefinite integral using Heronian mean and we derive Heronian mean Newton's method. At the end we do some numerical experiments for Mean Based Newton's method.

2. Preliminary Results

Definition 1 (See [2]) A sequence of iterates $\{x_n : n \geq 0\}$ is said to converge with order $p \geq 1$ to a point α if

$$|\alpha - x_{n+1}| \leq c\,|\alpha - x_n|, \qquad n \geq 0,$$

for some $c > 0$. If $p = 1$, the sequence is said to converge linearly to α. In that case, we require $c < 1$; the constant c is called the rate of linear convergence of x_n to α.

Let $e_n = x_n - \alpha$ be the error in the nth iterate of the method which produce the sequence $\{x_n\}$. Then the relation

$$e_{n+1} = ce_n^p + O(e_n^{p+1}) = O(e_n^p)$$

is called the error equation. The value of p is called the order of convergence and c is known as the asymptotic error constant of this method.

Definition 2 (See [10]) Let α be a root of the function f and suppose that x_{n-1}, x_n, x_{n+1} are closer to the root α. Then the computational order of convergence p can be approximated using the formula

$$p \approx \frac{\ln|(x_{n+1} - \alpha)/(x_n - \alpha)|}{\ln|((x_n - \alpha)/(x_{n-1} - \alpha))|} \tag{2}$$

3 Newton's Methods Based on Variety of Means

Newton's method (NM) for computing the root a of the nonlinear equation $f(x) = 0$ is to start with initial estimate x_0 sufficiently close to the root α and to use the one point iteration

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \tag{3}$$

where x_n is the nth approximation of α. We may also view x_{n+1} as the root of the two-term Taylor expansion or linear model of f about x_n [7],

$$M(x) = f(x_n) + f'(x_n)(x - x_n)$$

By integrating by part this local model can be viewed as the following obvious identity[4]

$$f(x) = f(x_n) - \int_{x_n}^{x} f'(s)ds \tag{4}$$

Weerakoon and Fernando [10] approximate the indefinite integral involved in (4) by the trapezoidal rule,

$$\int_{x_n}^{x} f'(s)ds = \left(\frac{f'(x_n) + f'(x)}{2}\right)(x - x_n) \tag{5}$$

and then by some algebra they end up with the following method

$$x_{n+1} = x_n - \frac{2f(x_n)}{f'(x_n) + f'(x^*_{n+1})}, \qquad n = 0,1,2,\cdots, \tag{6}$$

$$x^*_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}. \tag{7}$$

They prove that the method is a third order convergence. The way they choose x^*_{n+1} as in (7) was introduced for the first time by Wall [11].

Ozban [8] replaced arithmetic mean with harmonic mean in (5), that is

$$\int_{x_n}^{x} f'(s)ds = \left(\frac{2f'(x_n)f'(x)}{f'(x_n) + f'(x)}\right)(x - x_n)$$

He obtain Harmonic mean Newton's method (HN)

$$x_{n+1} = x_n - \frac{f(x_n)(f'(x_n) + f'(x^*_{n+1}))}{2f'(x_n)f'(x^*_{n+1})}, \qquad n = 0,1,2,\cdots, \tag{8}$$

$$x^*_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}. \tag{9}$$

Similarly Lukic and Ralevic [9] use geometric mean instead of arithmetic mean in (4), the find a formula for Geometric mean Newton's method (GM), as follows:

$$x_{n+1} = x_n - \frac{f(x_n)}{\text{sign}(f(x_0))\sqrt{f'(x_n)f'(x^*_{n+1})}}, \qquad n = 0,1,2,\cdots, \tag{10}$$

$$x^*_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}. \tag{11}$$

They show that this formula is of order three for simple root and linear for multiple roots.

Now if we substitute heronian mean to replace arithmetic mean in (4), we obtain

$$\int_{x_n}^{x} f'(s)ds = \left(\frac{f'(x_n) + f'(x) + \sqrt{f'(x_n)f'(x)}}{3}\right)(x - x_n). \tag{12}$$

On substituting (12) into (3), taking the next iteration of local model, and simplifying the resulting equations, we obtain

$$x_{n+1} = x_n - \frac{3f(x_n)}{f'(x_n) + f'(x_{n+1}) + \sqrt{f'(x_n)f'(x_{n+1})}}, \tag{13}$$

From (13) we propose Heronian mean Newton's method (HeM) as follows:

$$x_{n+1} = x_n - \frac{3f(x_n)}{f'(x_n) + f'(x_{n+1}^*) + \text{sign}(f(x_0))\sqrt{f'(x_n)f'(x_{n+1}^*)}}, \quad n = 0,1,2,\cdots, \tag{14}$$

$$x_{n+1}^* = x_n - \frac{f(x_n)}{f'(x_n)}. \tag{15}$$

4 Asymptotic Error Analysis

Theorem 1 Let $f: D \subseteq R \to R$ for an open interval *D*. Assume that *f* has first, second and third derivative in the interval *D* and *f* has a simple root $\alpha \in D$. If x_0 is sufficiently close to α, then the new method defined by (13) and (14) satisfies the following error equation:

$$e_{n+1} = \left(-\frac{1}{2}c_3 - \frac{5}{6}c_2^2\right)e_n^3 + O\left(e_n^4\right)$$

Where $e_n = x_n - \alpha$ and

$$c_j = \frac{\left(\frac{1}{j!}\right)f^{(j)}(\alpha)}{f'(\alpha)}, j=2,3$$

Proof:

We follow the technical proof from [10]. Let α be a simple root of $f(x) = 0$, and $x_n = \alpha + e_n$. We expand $f(x_n)$ about $x_n = \alpha$ using Taylor expansion, that is

$$f(x_n) = f(\alpha) + f'(\alpha)(x_n - \alpha) + \frac{1}{2!}f''(\alpha)(x_n - \alpha)^2 + \frac{1}{3!}f'''(\alpha)(x_n - \alpha)^3 + O\left((x_n - \alpha)^4\right).$$
$$= f'(\alpha)\left(e_n + c_2e_n^2 + c_3e_n^2 + O(e_n^4)\right) \tag{16}$$

Moreover we have

$$f'(x_n) = f'(\alpha) + \frac{1}{2!}f''(\alpha)(x_n - \alpha) + \frac{1}{3!}f'''(\alpha)(x_n - \alpha)^2 + O\left((x_n - \alpha)^3\right).$$
$$= f'(\alpha)(1 + 2c_2e_n + 3c_3e_n^2 + O(e_n^3)) \tag{17}$$

Computing $f(x_n)/f'(x_n)$ from (16) and (17), and recalling

$$(1+x)(1 - x + x^2 - x^3 + \cdots + O(x^n) = 1 + O(x^n) \tag{18}$$

we obtain

$$\frac{f(x_n)}{f'(x_n)} = \frac{f'(\alpha)(e_n + c_2e_n^2 + c_3e_n^2 + O(e_n^4))}{f'(\alpha)(1 + 2c_2e_n + 3c_3e_n^2 + O(e_n^3))}$$
$$= (e_n + c_2e_n^2 + c_3e_n^2 + O(e_n^4))(1 + 2c_2e_n + 3c_3e_n^2 + O(e_n^3))^{-1}$$
$$= e_n - c_2e_n^2 + 2(c_2^2 - c_3)e_n^3 + O(e_n^4) \tag{19}$$

On substituting (19) into (15) yields

$$x_{n+1}^* - \alpha = c_2e_n^2 - 2(c_2^2 - c_3)e_n^3 + O(e_n^4). \tag{20}$$

Expanding $f'(x_{n+1}^*)$ about $x = \alpha$, noting (20) and simplifying, we end up with

$$f'(x_{n+1}^*) = f'(\alpha)\left(1 + 2c_2^2e_n^2 - 4(c_2^3 - c_3c_2)e_n^3 + O(e_n^3)\right). \tag{21}$$

Now, using (17) and (21), we compute $f'(x_n)f'(x_{n+1}^*)$, that is

$$f'(x_n)f'(x_{n+1}^*) = \left(f'(\alpha)\right)^2\left(1 + 2c_2e_n + (2c_2^2 + 3c_3)e_n^2 + 4c_2c_3e_n^3 + O(e_n^4)\right). \tag{22}$$

Recalling

$$(1+r)^{1/2} = 1 + \frac{1}{2}r - \frac{1}{8}r^2 + \frac{1}{16}r^3 + O(r^4),$$

we have

$$\operatorname{sign}(f(x_0))\sqrt{f'(x_n)f'(x^*_{n+1})} = f'(\alpha)\left(1+2c_2e_n+(2c_2^2+3c_3)e_n^2+4c_2c_3e_n^3+O(e_n^4)\right)^{1/2}.$$

$$= f'(\alpha)\left(1+c_2e_n+\frac{1}{2}(c_2^2+3c_3)e_n^2+\frac{1}{2}(c_2c_3-c_2^3)e_n^3+O(e_n^4)\right). \quad (23)$$

Computing $f(x_n)+f'(x^*_{n+1})$ we obtain

$$f(x_n)+f'(x^*_{n+1}) = f'(\alpha)\left(2+2c_2e_n+(2c_2^2+3c_3)e_n^2-4(c_2^3-c_2c_3)e_n^3+O(e_n^4)\right) \quad (24)$$

Using (16), (23) and (24), recalling (18), and simplifying we obtain

$$\frac{3f(x_n)}{f'(x_n)+f'(x^*_{n+1})+\operatorname{sign}(f(x_0))\sqrt{f'(x_n)f'(x^*_{n+1})}}$$

$$= \frac{f'(\alpha)\left(e_n+c_2e_n^2+c_3e_n^3+O(e_n^4)\right)}{f'(\alpha)\left(3+3c_2e_n+\frac{1}{2}(5c_2^2+9c_3)e_n^2+\frac{1}{2}(9c_2c_3-9c_2^3)e_n^3+O(e_n^4)\right)}.$$

$$= e_n+\left(-\frac{1}{2}c_3-\frac{5}{6}c_2^2\right)e_n^3+O\left(e_n^4\right) \quad (25)$$

On substituting (25) to (14) we have

$$e_{n+1} = \left(-\frac{1}{2}c_3-\frac{5}{6}c_2^2\right)e_n^3+O\left(e_n^4\right)$$

and this ends the proof.□

5 Comparison

5.1 Analytical Comparisons

For analytical comparisons we look into the cost of function evaluation for the methods, as in Table 1. From this Table, we see that all methods need only one function evaluation. The Mean-Based Newton's Methods (MBN) need an expense in the derivative of f. From this point, we can say that all MBN are comparable in term of the functional evaluations. However if we look into the addition and multiplication costs, the GM is fewer than the other MBN.

Table 1. Comparisons of the computational costs of the Mean-Based Newton's Methods

Method	Cost of			
	Addition/subtraction	Multiplication/division	$f(x)$	$f'(x)$
NM	1	1	1	1
AM	3	3	1	2
HN	3	5	1	2
GM	2	3	1	2
HeM	4	4	1	2

5.2 Numerical Experiments

The MBN and NM are tested using some functions, which have been used in [10] and [9]. We compare the number of iterations for each method by varying some initial values. We also compute the computational order of convergence of the methods (2) for each initial value. We stop the program using the following criteria

$$|x_{n+1}-x_n| < \varepsilon |x_{n+1}|$$
$$|f(x_{n+1})| < \varepsilon$$

where $\varepsilon = 1.0e-12$. All computations are done using Matlab on Windows PC with Intel Processor at 2.4 GHz. The computational results are given in Table 2.

From Table 2, we see that the results of our proposed method (HeM) are comparable with the other MBN Methods. This is in agreement with the theoretical results as stated in Theorem 3. The HeM also has the same characteristics as other MBN Methods, that is:

1. third order of convergence for simple root,
2. does not require the computation of second or higher order derivatives,

Table 2. Comparisons of the number of iterations of the Mean-Based Newton's Methods

$f(x)$	x_0	Number of iterations					COC				
		NM	AM	HM	GM	HeM	NM	AM	HM	GM	HeM
x^3+4x^2-10	0.5	6	4	3	4	4	2.0	3.01	3.04	ND	ND
	1.0	5	3	3	3	3	ND	ND	ND	ND	ND
	2.0	5	3	3	3	3	ND	3.0	ND	ND	ND
$x\exp(x^2)-\sin^2 x$ $+3\cos(x)+5.0$	1.0	7	55	5	9	31	2.0	3.0	ND	ND	ND
	2.0	8	5	5	5	5	ND	3.01	ND	ND	ND
	3.0	13	9	8	8	9	2	ND	ND	3.0	ND
$\sin^2 x-x^2+1$	0.1	15	45	8	5	32	0.86	1.02	0.8	ND	ND
	1.0	5	4	3	3	3	2	0.9	ND	3.01	ND
	3.0	6	3	3	3	3	ND	ND	2.58	3.01	2.19
$2x\exp(-20)+1$ $-2\exp(-20x)$	0.1	7	5	4	5	5	2.0	2.18	3.01	1.13	1.81
	0.0	5	3	3	3	3	1.85	2.99	ND	3.0	2.99
	0.1	7	5	3	4	5	1.7	3.02	3.0	3.0	2.02
$\exp\left(\frac{1}{x}-25\right)-1$	0.01	81	55	44	49	53	1.85	2.98	ND	2.99	2.99
	0.03	13	9	7	8	9	2.0	ND	3.02	3.0	ND
	0.042	7	5	3	4	5	ND	3.02	3.0	3.01	ND
$\exp(x^2+7x-30)-1$	2.85	10	14	4	6	12	ND	ND	3.01	ND	0.57
	3.5	11	8	7	7	8	2.0	ND	ND	ND	ND
	6.5	63	43	34	38	41	ND	ND	ND	2.94	3.01
$(x-1.0)^3-1$	2.5	5	4	3	3	4	2.0	ND	ND	2.99	ND
	4.0	7	5	4	4	5	2.02	ND	ND	2.99	ND
	1.5	7	5	4	4	4	ND	ND	ND	ND	3.01
	1.0	10	8	6	11	5	ND	ND	ND	ND	ND
	0.1	12	6	6	5	4	ND	ND	3.03	ND	ND

NM - Newton's Method
AM - Arithmetic Mean Newton's Method
HM - Harmonic Mean Newton's Method
COC -Computational Order of Convergence
GM - Geometric Mean Newton's Method
HeM - Heronian Mean Newton's Method
ND - Not Defined

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