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Developing A Direct Search Algorithm for Solving The Capacitated Open Vehicle Routing Problem

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Abstract. In open vehicle routing problems, the vehicle are not required to return to the depot after completing service. In this paper, we present the first exact optimization algorithm for the open version of the well-known capacited vehicle routing problem (CVRP). The strategy of releasing nonbasic variables from their bounds, combined with the "active constraint" method and the notion of supernasics, has been developed for efficienty requirments, this strategy is used to force the appropriate non-integer basic variables to move to their neighbourhood integer points. A study of criteria fr choosing a nonbasic variable to work with in the inegerizing strategy has also been made. Succesful implementation of these algorithms was achieved on various test problem.

Key words: vehicle routing, superbasics, neighbourhood search

1. Introduction

In the classical version of Vehicle Routing Problems (VRPs), the vehicles are required to return to depot after completing service (see for example Toth & Vigo, 2002). In open VRPs, however, the vehicles need not do so. As a result, the vehicles routes are not closed path but open ones, starting at the depot and ending at one of the customers. Figure 1, which shows the optimal solution to both the open version of a VRP can be quite different from that for the closed version. (Throughout this paper, the depot is represented by a white square and the customers by black circle).

At the first sight, having open routes instead of closed ones looks like a minor modification. Indeed, if travel cost are asymmetric, there is essentially no difference between the open and closed version: to transform the open version into the closed one, it suffices to set the cost to zero for traveling from any customer to the depot. However, if travel costs are symmetric, things are more subtle. Indeed, we prove in the next section that, somewhat suprisingly, the open version turns out to be more general than closed one, in the sense that any closed VRP on n customers can be transformed into an open VRP on n customers, but there is no transformation in the reverse direction.

Moreover, there are many practical application in which open VRPs naturally arise. This happens for example when a company does not own a vehicle fleet and all its deliveries from a central depot are undertaken by hired vehicles that are not obliged to return to the depot . In such situations, the cost of the distribution may be proportional to the distance travelled while loaded. A practical case study of this type is describe in Tarantilis et al. (2004, 2005). The same model can also be used for pick-ups, where vehicle start empty at any customer and must pick up the demands of each customer on their route and deliver them to the depot.

There are also applications where the vehicles start at the depot, deliver to a set of customers and then are required to visit the customers in reverse order, picking up items that are required to be backhauled to the depot. If, each for customers, the pick-up demand is no larger than the delivery demand, then an open VRP model can be used. An application of this type for in an express courier is mentioned by Schrage (1981) in an early article describing features of practical routing problems.

Two further areas of application are described by Fu et al. (2005). The first involves the planning of train services, starting or ending at the Channel Tunnel. The second involves planning a set of a school bus routes where in the morning pupils are picked up at various locations and brought to school, and in the afternoon, the routes are reserved to take pupils home. Bodin et al. (1983) includes a description of a problem of express airmail distribution in the USA, that is essentially an open pick-up and delivery VRP with capacity constraints and time windows.

Open VRPs are easily seen to be strongly NP-hard by reduction from the Hamiltonian path problem. Research on open VRPs has therefore up to now concentrated on devising effective heuristics for solving them. For the version involving only capacity constraints, Sariklis and Powel (2000) presented a two-phase heuristic involving minimum spanning trees, Tarantilis et al. (2004) present a population-based heuristic , and Tarantilis et al. (2005) present a heuristic of the threshold-accepting type. For a more general variant involving both capacity and route-lenght constraints, Brand ao (2004) and Fu et al. (2005, 2006) describe tabu search heuristics, Li et al. (2006) present a record-to-record travel heuristic, and Pisinger & Ropke (2006) present an adaptive neighborhood search heuristic . Heuristic have also been devised for open VRPs with other kinds of constrains; see for example Repoussis et al. (2006) and Aksen et al. (2006).

In this paper we present the first exact optimization algorithm for the *Capacitated Open Vehicle Routing Problem* (COVRP), which is defined as follows. A complete undirected graph G=(V, E) is given, with V={0, . . . ,n}. Vertex 0 represents the depot, the other vertices represent customers. The cost of travel from vertex i to vertex j is

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denoted by c_{ij} and we assume cost are symmetric, $c_{ij} = c_{ji}$. A fleet of K identical Q. Each customer must be serviced by a single vehicle and no vehicle may serve a set of customers whose total demand exceeds its capacity. Each vehicle route must start at the depot and end at the last customer it serves. The objective is to define the set of vehicle routes that minimizes the total costs.

As we will show, our algorithm is capable of solving small-to medium-size instances to optimality, and providing useful lower bounds for larger instances. It can also be easily adapted to handle some other variants of the COVRP, such as variants with a free vehicle fleet size, or with a fixed cost associated with the use of a vehicle. The structure of the remainder of the paper is as follows. In the next section, we give an integer programming formulation of the COVRP and present some valid inequalities. It will be seen that more complex inequalities are needed for the open version than for the closed version. Then, in the following section, we describe the ingredients of our branchand-cut algorithm. Extensive computational result are given in the following section, which enable us for the first time to assess the quality of existing heuristic methods, and to compare the relative difficulty of open and closed version of the same problem. Some concluding remarks are given in the final section.

2. Formulation and Valid Inequality

2.1 Formulation

The COVRP is clearly a special case of the asymmetric CVRP (ACVRP), in which, for any i, j 2 V, cij is permitted to be different from cji. Hence, it would be possible to use any integer programming formulation of the ACVRP (for example, that of Fischetti et al. 1994) to solve the COVRP. However, this would mean that effectively we were treating each (undirected) edge as two (directed) arcs, which would lead to a formulation of the COVRP with twice as many variables as our formulation of the CVRP. This seems unnecessary, given that, in the COVRP, cij=cji when i and j are customers.

Amore parsimonious formulation of the COVRP can be obtained by modifying the standard formulation of the CVRP. To explain this clearly, it is helpful to recall the following details of the CVRP formulation.

Let $V_c = v\backslash\{0\}$ denote the set of customers. Given a set $S \subseteq V_c$, let $q(S)$ denote $\sum_{i\in S} q_i$, $\delta(S)$ denote the set pf edges in G with both end-vertices in S, and k(S) denote $\lceil q(S)/Q \rceil$. Obviously, k(S) is a lower bound on the minimum number of vehicles required to serve the customers in S. Let x_{ij} represent the number of times a vehicles travels between vertices i and j. (Because the problem is undirected, x_{ij} and x_{ji} represent the same variables.) Finally, given an arbitrary $F \subseteq E, x(F)$ will denote $\sum_{e\in F} x_e$. Then the standard (so-called two-index) formulation of the CVRP is (Laporte, Nobert & Desrochers, 1985):

$$Minimize \quad \sum_{e\in E} c_e x_e$$

subject to:

$$x(\delta\{i\})) = 2 \qquad (i = 1, \dots, n) \qquad (1)$$
$$x(\delta(S)) \geq 2k(S) \qquad (S \subseteq V_c |S| \geq 2) \qquad (2)$$
$$x(\delta\{0\})) = 2K \qquad (3)$$
$$x_{ij} \in \{0,1\} \qquad (1 \leq i < j \leq n) \qquad (4)$$
$$x_{0j} \in \{0,1,2\} \qquad (i = 1, \dots, n \quad) \qquad (5)$$

The degree *equation* (1) ensure that customers are visited exactly once. The *rounded capacity inequalities* (2) impose the vehicle capacity restrictions and also ensure that the route are connected. They can be re-written, using the degree equations, in the Iterative form

$$x(E(S) \leq |S| - k(S) \qquad (6)$$

The equation (3) ensure that exactly K vehicles are used. Finally, constraints (4) and (5) are the integrality conditions. Note that the variables x_{0i} are permitted to take the value 2, to allow routes in which a vehicle serve a single customer.

To adapt this formulation to the COVRP, we simply treat each edge incident on the depot as a pair of directed arcs, as follows. For each $i \in V_c$, instead of defining the undirected variable x_{0i}, we define the binary variable y_{0i}, which takes the value 1 if and only if vehicle travels directly from the depot to i,and the variable y_{0i}, which takes the value 1 if only f a vehicle travels from i to the depot. We also use the notation $y^-(S) = \sum_{i\in S} y_{0i}$ and $y + (S) = \sum_{i\in s} y_{0i}$. Finally, for any $S \subset V_c$ we use the notation $\bar{S} = V_c \backslash S$ and $\bar{\delta}(S) = ((i,J\}: i \in S, j \in \bar{S}\}$.

If weadapt the above formulation to the COVRP in straightforward manner, we obtain the following integer program:

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Minimize $\sum e \in E(V_c)cexe + \sum_{i \in V_c} c_{0i} y_{0i}$

subject to:

$x(\bar{\delta}(i) + y_{0i} + y_{i0} = 2$ $\quad (i = 1, \ldots, n) \quad (7)$

$x\left(\bar{\delta}(S)\right) + y^{-}(S) + y^{+}(S) \geq 2k(S)$ $\quad (S \subseteq V_c |S| \geq 2) \quad (8)$

$y^{-}(V_c) = k(9)$

$y^{+}(V_c) = k(10)$

$x_{ij} \in \{0,1\}$ $\quad (1 \leq i < j \leq n) \quad (9)$

$y_{0i}, y_{i0} \in \{0,1\}$ $\quad (i = 1, \ldots, n \quad) \quad (10)$

The constraints (7), (8) are straightforward adaption of degree equations and rounded capacity inequalities, respectively. The inequalities (8) can again be re-written in the form $x(E(S) \leq |S| - k(S)$. The constraints (9) and (10) ensure that exactly K vehicles leave and enter the depot,, Finally, constraints (11) and (12) ensure that all variable are binary. (There is no longer any need to allow any variables to take value 2.)

Perhaps surprisingly, the above integer program does not represent a valid formulation for the COVRP. Figure 2 shows a solution to the above integer program for a small COVRP instance with n=4, which does not represent a valid solution to the COVRP.

To prevent invalid solutions of this kind, t is necessary to add the following constraints to the formulation:

$$x\left(\bar{\delta}(S)\right) + y^{+}(S) \geq y^{-}(S) \qquad (S \subseteq V_c |S| \geq 2. \qquad (13)$$

We call these constraints balancing inequalities. The fact that they are valid, and sufficient to ensure feasibility, follows from the conditions of Ford and Fulkerson (1962) for a mixed graph to be Eulerian. (Some similar inequalities were introduced by Nobert and Picard, 1996, in the context of the so-called (Mixed Chinese Postman Problem.)

The invalid solution above, for example, violates the balancing inequality with S = {1,2} ,which takes the form

$$x_{13} + x_{14} + x_{23} + x_{24} + y_{10} + y_{20} \geq y_{01} + y_{02}.$$

It turns out that, once the balancing inequalities have been added to the formulation, the equation (9) is redundant.

Note that there are an exponential number of balancing inequalites. The need for balancing inequalities, which have no counterpart for the standard CVRP , suggest that COVRP is a more complex problem then the CVRP. This is confirmed by the following definition and proposition.

Definition 1 The partially asymetric CVRP(PACVRP) is the generelization of the CVRP in which the cost of travel c_{0i} is permitted to be different from c_{i0}

Obviosly, the PACVRP is the intermediete in generality between the CVRP and CVRP. The following result is less obvios.

Proposition 1 The COVRP and the PACVRP are equivalent.

Proof: Any COVRP instance is clearly a PACVRP instance. Now, suppose we are given a PACRVP and asymmetric travel costs cij for all $\{i, j\} \in E(V_c)$ and asymetric travel costs $c_{0i},\ c_{i0}$ for all $i \in V_c$. Now, let M be an arbitrary constant. Due to the presence of the degree equation (7), (9) dan (10), the optimal solution to the PACRVP is unchanged if we replace the original travel costs c_{ij} with modified cost c_{0ij}, defined as follows:

- for all $\{i, j\} \in E(V_c), c'_{ij} = c_{ij} - c_{io} - c_{j0} + M,$
- for all customers i, $c'_{0i} = c_{0i}$ and $c'_{0i} = 0.$

If we choose M appropriatelly, the transformed costs c'_{ij} will be on non-negative. (An appropriate value of M is max1_i<j_n{$c_{i0} + c_{j0} - c_{ij}$}.) Since the costs c0i0 are now zero, we have a COVRP instance.

The algorithm we propose in this paper can therefore be used to solve instance of the PACVRP.

We remark that an alternative integer programming for the COVRP can be obtained by eleminating the variables yi0, which do not appear in the objective function, via the equation (7). Although the resulting formulation has and fewer variables, it is harder to understand and analyse and, more importantly, has a higher density (proportion of nonzeroes), which is unattractive from a computational point of view. For these reasons, we prefer to use the original formulation.

Symmetric inequalities from a poly of view, the COVRP is similar to the CVRP. The following proposition, which is trivial to prove, shows that any valid inequality for the CVRP yields a valid inequality for the COVRP.

Proposition 2 $Let\ \sum_{0 \leq i < j \leq n} \alpha_{ij} x_{ij}\ be\ valid\ for\ the\ CVRP.$

$Then\ \sum_{1 \leq i < j \leq n} \alpha_{ij} x_{ij} + \sum_{i=1}^{n} \alpha_{oj}(y_{0i} + y_{i0}) \leq \beta\ is\ valid for\ the\ COVRP.$

We call inequalities obtained in this way symmetric. A simple example of a class of symmetric inequalities is given by the inequalities (8), which are clearly the symmetric version of the rounded capacity inequalities (2). Other valid inequalities for the CVRP include, for *example, the homogeneous multistar and partial multistar, generalized large multisar, framed capacity, strengthened comb, and hypotour inequalities.* See Augerat (1995), Letchford et al.

(2002), Lysgaard et al. (2004) and Naddef & Rinaldi (2002) for details. From Proposition 2, these all have valid counterparts for the COVRP.

2.2 Asymmetric inequalities

We say that a valid inequalities $\alpha x + \beta x \geq \gamma$ for the COVRP is asymmetric if there exists at least one $i \in V$, such that $\beta_{0i} \neq \beta_{i0}$. The existence and necessity of the balancing inequalities shows that there exist non-redundant asymmetric inequalities. This should be excepted, since to COVRP is a generalization of the CVRP.

Using the degree equation, it possible to write the balancing inequalities in a variety of forms. In particular, the balancing inequalities for S equivalent to $x\left(\bar{\delta}(\bar{S})\right) + y^-(\bar{S}) \geq y^+(\bar{S})$. Therefore, there is no need for a 'reversed' form of the balancing inequalities, of the form $x\left(\bar{\delta}(\bar{S})\right) + y^-(\bar{S}) \geq y^+(\bar{S})$, since this is equivalent to the balancing inequalities on ¯S.

Unfortunately, the addition of all symmetric inequalities to the bounds degree equation and balancing inequalities still does not give a complete description of the COVRP polyhedron. Suppose n=6, Q=5 and $q_i = 1$ for i = 1, . . . , 6, and consider the fractional point displayed in Figure 3. (The dotted line represent edges whose variables have value 1/2.) It is easy to check that it satisfies the bounds, degree equations and balancing inequalities. Moreover, it satisfies all symmetric inequalities. To see this, note that, if we replace the directed arcs with undirected edges, the resulting fractional point for the CVRP is a convex combination of the two integral points displayed in Figure 4.

The fractional point displayed in Figure 3 can be cut off by the inequality $x_{12} + x_{15} + x_{25} + x_{56} + y_{01} + y_{20} \leq 4$. This inequality, which is easily seen to be valid for the COVRP, is a special case of a class of inequalities which we call mixed strengthened comb inequalities. These inequalities are presented in the following theorem.

Theorem 1 *Let $H \in V_c$ (the handle)* and $T_i, \dots, T_t \subset V$ *(the teeth)* be such that:

- *every tooth properly intersects with the handle, i.e., $T_i \subset H$ end $T_i \backslash H$ are non-empty for all i;*
- *if any pair of teeth intersect, then either all vertices in the intersection lie in the andle or all lie outside, i.e. for $1 \leq i < j \leq t$, either $Ti \cap Tj \subset H$ or $Ti \cap Tj \cap H = \phi$*

Moreover, let the index set {1, . . . , t} be partitioned into sets N,D,S,R such that $0 \notin T_i$ for all $i \in N$ and $0 \in T_i$ for all $i \in D \cup S \cup R$. We call Ti a normal tooth if $i \in N \cup D$, a sending tooth if $i \in S$ and a receiving tooth if $i \in R$. Now, for any tooth T_i, we define:

$$\gamma(T_i) = \begin{pmatrix} k(T_i + k(T_i \cap H) + k(T_i \backslash H) & if\ i \in N \\ k(V \backslash T_i) + k(T_i \cap H) + k\big(V(T_i \backslash H)\big)(Ti) = 8 > & if\ i \in D \\ k(T_i \cap K) + 2 & if\ i \in S \cup R \end{pmatrix}$$

Then, if $\sum_{i=1}^{n} \gamma(T_i)$ is odd, the mixed comb inequality

$$x\big(E(H)\big) + \sum_{i \in N} x\big(E(T_i)\big) + \sum_{i \in D \cup S \cup R} x(E(Ti\{0\})) + \sum_{i \in D \cup S} y^+(Ti\{0\}) + \sum_{i \in D \cup R} y^-(Ti\{0\}) \leq |H| + \sum_{i=1}^{t} |T_i| + |D|(K-1) - [\textstyle\sum_{i=1}^{t} \gamma(Ti)/2] \quad (14)$$

is valid for the COVRP.

The proof of this theorem is in Appendix.

The mixed strengthened comb inequalities reduce to ordinary strengthened comb inequalities (Lysgaard et al., 2004) when there are no sending and receiving teeth. The fractional point displayed in Figure 3 violates mixed comb inequality with handle H={1, 2, 5}, normal tooth {5, 6}, receiving tooth {0,1} and sending tooth {0,2}.

For many instance of the CVRP or COVRP, the number of vehicles K is fixed at the minimum possible, which often equals $k(V_c)$. In such a case, a lower bound on the amount which must be loaded onto any vehicle in any fesible solution is $q_{min} = q(V_c) - Q(K-1) \geq 0$. It is easy to show that it is never worthwhile having a vertex set as a sending or receiving tooth unless the total demand of the set it least qmin, and that it is not worthwhile having a customer set T as a normal tooth if $k(T) = k(T \cap H) + K(T \backslash H)$.

3. The Algorithm

After solving the relaxed problem, the procedure for searching a suboptimal but integer-feasible solution from an optimal continuous solution can be described as follows.

Let

$x = [x] + f, \quad 0 \leq f \leq 1$

be the (continuous) solution of the relaxed problem, [x] is the integer component of non-integer variable x and f is the fractional component.

Step1. Get row i^* the smallest integer infeasibility, such that

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$\delta_{i^*} = \min\{f_i,\ 1 - f_i\}$

Step 2. Calculate

$v_{i^*}^T = e_{i^*}^T B^{-1}$

This ia a pricing operation

Step 3. Calculate $\sigma_{ij} = v_{i^*}^T a_j$

With $j$corresponds to $\min_j \left\{ \left| \frac{d_j}{\sigma_{ij}} \right| \right\}$

I. *For nonbasic j at lower bound*

If $\sigma_{ij} < 0$ and $\delta_{i^*} = f_i$ calculate $\Delta = \frac{(1-\delta_{i^*})}{-\sigma_{ij}}$

If $\sigma_{ij} > 0$ and $\delta_{i^*} = 1 - f_i$ calculate $\Delta = \frac{(1-\delta_{i^*})}{\sigma_{ij}}$

If $\sigma_{ij} < 0$ and $\delta_{i^*} = 1 - f_i$ calculate $\Delta = \frac{\delta_{i^*}}{-\sigma_{ij}}$

If $\sigma_{ij} > 0$ and $\delta_{i^*} = f_i$ calculate $\Delta = \frac{\delta_{i^*}}{\sigma_{ij}}$

II. *For nonbasic j at upper bound*

If $\sigma_{ij} < 0$ and $\delta_{i^*} = 1 - f_i$ calculate $\Delta = \frac{(1-\delta_{i^*})}{-\sigma_{ij}}$

If $\sigma_{ij} > 0$ and $\delta_{i^*} = f_i$ calculate $\Delta = \frac{(1-\delta_{i^*})}{\sigma_{ij}}$

If $\sigma_{ij} > 0$ and $\delta_{i^*} = 1 - f_i$ calculate $\Delta = \frac{\delta_{i^*}}{\sigma_{ij}}$

If $\sigma_{ij} < 0$ and $\delta_{i^*} = f_i$ calculate $\Delta = \frac{\delta_{i^*}}{-\sigma_{ij}}$

Otherwise go to next non-integer nonbasic or superbasic j (if available). Eventually the column $f^*$is to be increased form LB or decreased from UB. If none go to next i^*.

Step 4. Calculate

$a_{j^*} = B^{-1} a_{j^*}$

i.e. solve $B a_{j^*} = a_{j^*}$ for a_{j^*}.

Step 5. Ration test; there would be three possibilities for the basic variables in order to stay feasible due to the releasing of nonbasic j^* for its bounds.

If j^* lower bound

Let

$$A = \min_{i' \neq i^* | \alpha_{ij^*} > 0} \left\{ \frac{x_{B_{i'}} - l_{i'}}{\alpha_{ij^*}} \right\}$$

$$B = \min_{i' \neq i^* | \alpha_{ij^*} < 0} \left\{ \frac{u_{i^*} - x_{B_{i'}}}{-\alpha_{ij^*}} \right\}$$

$$C = \Delta$$

the maximum movement of j^* depends on:

$$\theta^* = \min(\mathrm{A}, \mathrm{B}, \mathrm{C})$$

If j^* upper bound

Let

$$A' = \min_{i' \neq i^* | \alpha_{ij^*} < 0} \left\{ \frac{x_{B_{i'}} - l_{i'}}{\alpha_{ij^*}} \right\}$$

$$B' = \min_{i' \neq i^* | \alpha_{ij^*} > 0} \left\{ \frac{u_{i^*} - x_{B_{i'}}}{-\alpha_{ij^*}} \right\}$$

$$C' = \Delta$$

the maximum movement of j^* depends on:

$$\theta^* = \min(A', \mathrm{B}', \mathrm{C}')$$

Step 6. Exchanging basis for the three possibles

1. If A or A'
   - $x_{B_{i'}}$, becomes nonbasic at lower bound $l_{i'}$
   - x_{j^*} becomes basic (replaces $x_{B_{i'}}$)
   - x_{i^*} stays basic (non-integer)

2. If B or B'

- $x_{B_{i'}}$, becomes nonbasic at upper bound $u_{i'}$
- x_{j^*} becomes basic (replaces $x_{B_{i'}}$)
- x_{i^*} stays basic (non-integer)

3. If C or C'

- x_{j^*} becomes basic (replaces x_{i^*})
- x_{i^*} becomes superbasic at integer-valued

repeat from step 1.

4. Conclusion

Although the COVRP appears to be a trivial variant of the standard CVRP, we seen that it is intermediate in generality between the CVRP and the ACVRP, As a result, some subtle modification are needed to adapt a branch-and-cut code for the CVRP to the COVRP. This include modifications to the formulation, additional classes of inequalities, and adjustments to the separation algorithms.

Our results show that small-to medium-scale instances of the COVRP are just as amenable to exact solution by branch-and-cut as their CVRP counter parts. In fact, if anything, the open version often appear to be slightly easier. Future research could include the incorporation of column generation, leading to a full branch-cut-and-price algorithm along the lines of the one presented in Fukasawa et al. (2006). This would no doubt lead to the solution of even more instances, especially those with small vehicle capacities and a large number of vehicles.

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Appendix

To show validity of the mixed comb inequalities, it is helpful to prove the following lemma.

Lemma 1 *For anyset S such that* $0 \in S$*, the following three inequalities are valid.*

$$x(E(S\{0\})) + y^{+}(S\backslash\backslash\{0\}) + y^{-}(S\{0\}) \leq |S| + K - k(V\backslash S) - 1$$
$$x(E(S\{0\})) + y^{+}(S\backslash\backslash\{0\}) \leq |S| - 1$$
$$x(E(S\{0\})) + y^{-}(S\backslash\backslash\{0\}) \leq |S| - 1$$

Proof: Due to the degree equations, the inequality (16) is equivalent to the capacity inequality on $V\backslash S$ and the inequalities (17) and (18) are equivalent to the balancing inequalities on $V\backslash S$ and $S\{0\}$, respectively.

Proof of Theorem 1: We follow the standard Chvatal-Gomory integer rounding argument. If we sum together the following inequalities:

the degree equations for all $i \in H$,
the inequality (6) on $H \cap T_i$ for $1 \leq i \leq t$,
the inequality (6) on T_i and $T_i\backslash H$ for $i \in N$,
the inequalities (16) for T_i and $T_i\backslash H$, for $i \in D$,
the inequalities (17) for T_i and $T_i\backslash H$, for $i \in S$, and
the inequalities (18) for T_i and $T_i\backslash H$, for $i \in R$,

we obtain (after some re-arranging):

$$2x(E(H)) + 2\sum_{i\in N} x(RE(T_i)) + 2\sum_{iD\cup S\cup R} x(E(T_i\{0\})) + 2\sum_{i\in D\cup S} y^{+}(T_i\{0\}) + 2\sum_{i\in\in D\cup R} y^{-}(T_i\backslash\{0\} + x(\delta(H)\backslash \cup_{i=1}^{t} E(T_i)) + y^{+}(H\backslash S_{i=1}^{t} T_i) + y^{-}(H\backslash S_{i=1}^{t} T_i) \leq 2|H| + 2\sum_{i=1}^{t}|T_i| + 2|D|(K-1) - \sum_{i=1}^{t}(T_i).$$

Dividing this inequality by two and rounding down yields the result.

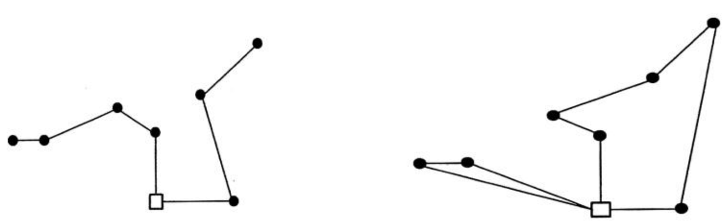

Figure 1 : Open closed routes with different clusterings of customers.

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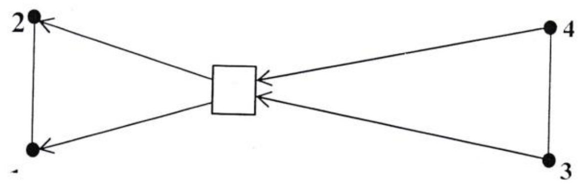

Figure 2 : Invalid integer solution

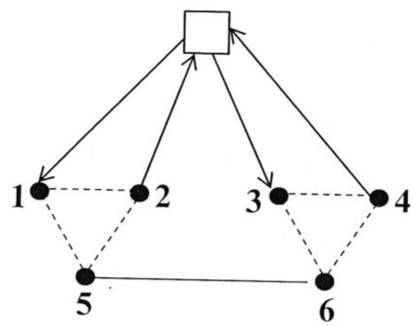

Figure 3 : Fractional point satisfying all balancing and symmetric inequalities

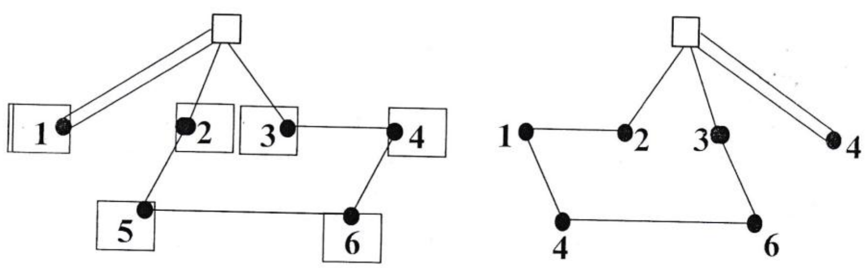

Figure 4 : Two feasible CVRP solutions.

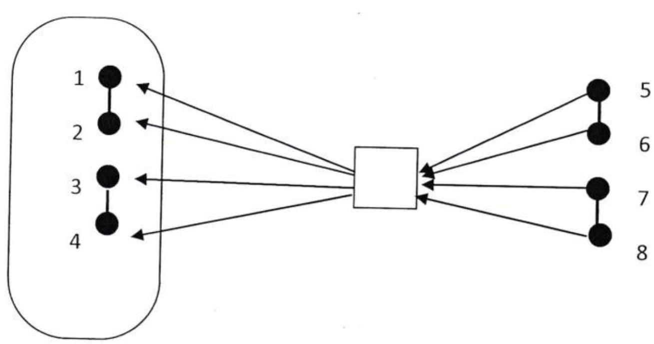

Figure 5 : Decomposing a balancing inequality.