A New Method for Dual Fully Fuzzy Linear Systems by use LU Factorizations of the Coefficient Matrix

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Abstract

In this paper we will discuss an alternative way to solve dual fully fuzzy linear system of the form $\widetilde{A} \otimes \widetilde{x} \oplus \widetilde{b} = \widetilde{C} \otimes \widetilde{x} \oplus \widetilde{d}$. *Next we crisp coefficient matrix* $\widetilde{A}$ *and* $\widetilde{C}$ *through form LU factorizations and we will construct a simple algorithm for the solution of these system. Finally we will illustrate our method by solving some examples.*

Keywords : Triangular fuzzy number, Fully fuzzy linear systems, Dual fuzzy linear system, LU Decompositions.

Abstrak

Dalam tulisan ini akan dibahas alternatif penyelesaian untuk sistem linear fuzzy dual penuh dalam bentuk $\widetilde{A} \otimes \widetilde{x} \oplus \widetilde{b} = \widetilde{C} \otimes \widetilde{x} \oplus \widetilde{d}$ *dengan mentransformasikannya ke dalam bentuk* $\widetilde{A} \otimes \widetilde{x} = \widetilde{C} \otimes \widetilde{x} \oplus \widetilde{f}$. *Selanjutnya kita perkecil matrik koefisien* $\widetilde{A}$ *dan* $\widetilde{C}$ *dalam bentuk faktorisasi LU dan akan kita kontruksi algoritma sederhana untuk penyelesaian system tersebut. Dibagian akhir diberikan contoh penyelesaian sebagai ilustrasi dari metoda yang diberikan.*

Kata kunci **:** *Bilangan fuzzy segitiga, Sistem linear fuzzy penuh, Sistem linear fuzzy dual penuh, Dekomposisi LU*

1. Introduction

We were first attracted by the problem of fuzzy linear systems (FLS) as in Friedman *et al.,* (1998). Various researchers, such as Dehghan *et al.,* (2006a; 2006b; 2007) and Nasseri *et al.*, (2008), have further extended the problem to the fully fuzzy linear systems (FFLS) of the form $\widetilde{A} \otimes \widetilde{x} = \widetilde{b}$ where $\widetilde{A}$ is a matrix of triangular fuzzy numbers and $\widetilde{b}$ is a fuzzy vector with triangular fuzzy numbers as its components. On the other hand, Allahviranloo et al., (22008) have studied the similar systems of the form $\widetilde{A} \otimes \widetilde{x} = \widetilde{b}$, where $\widetilde{A}$ is a matrix of fuzzy numbers in a parametric form and $\widetilde{b}$ is a fuzzy vector with fuzzy number in parametric form as its components.

Another form of fuzzy linear system $\widetilde{A} \otimes \widetilde{x} \oplus \widetilde{b} = \widetilde{C} \otimes \widetilde{x} \oplus \widetilde{d}$ is the dual fully fuzzy linear system (DFFLS). In particular, for real matrices $\widetilde{A}$ and $\widetilde{C}$, and fuzzy numbers in the parametric form $\widetilde{b}$ and $\widetilde{d}$, also in general, it is the dual fuzzy linear system (DFLS) of the form $A\widetilde{x} \oplus \widetilde{b} = C\widetilde{x} \oplus \widetilde{d}$, as been discussed by Abbasbandy *et al.,* (2008); Ezzati, (2008); and Mosleh *et al.*, (2008). The DFFLS where $\widetilde{A}$ is a fuzzy matrix of triangular fuzzy numbers or the parametric fuzzy numbers have not been any much attention yet, besides Mosleh *at al.,*

The numerical example given by Mosleh *et al.,* (2007) is of the form

$$\begin{cases} (1,2,3)\widetilde{x}_1 + (4,6,9)\widetilde{x}_2 + (1,3,4) \\ \quad = (0,1,3)\widetilde{x}_1 + (5,6,8)\widetilde{x}_2 + (0,1,7) \\ (5,6,8)\widetilde{x}_1 + (3,5,6)\widetilde{x}_2 + (0,7,8) \\ \quad = (1,4,5)\widetilde{x}_1 + (2,3,4)\widetilde{x}_2 + (8,9,12). \end{cases}$$

Its solution in the parametric form is

$$\underline{x}_1(r) = r - 3, \overline{x}_1(r) = -1 - r$$

$$\underline{x}_2(r) = 2r + 1, \overline{x}_2(r) = 4 - r$$

which are transformed as triangular fuzzy numbers $\widetilde{x}_1$ = (-3, -2, -1) and $\widetilde{x}_2$ = (1, 3, 4). However, we observed that $\widetilde{x}_1$ and $\widetilde{x}_2$ are not compatible solution when we substitute into the original equation.

Hence in this paper, we will discuss a new direct method for solving DFFLS $\widetilde{A} \otimes \widetilde{x} \oplus \widetilde{b} = \widetilde{C} \otimes \widetilde{x} \oplus \widetilde{d}$, with fuzzy coefficient matrix of triangular fuzzy numbers and also the parametric fuzzy numbers. We will illustrate with an example and validate that the solution fit the original systems.

2. Preliminaries

Some of the basic definitions of fuzzy number have been in Dehghan, (2006); Friedman, (1998); and Ma, (2000). We recall some definitions :

***Definition* 2.1**. A fuzzy number is a fuzzy set $\widetilde{u} : R \rightarrow [0,1]$ which satisfies the following:

a. $\tilde{u}$ is upper semicontinuous;
b. $\tilde{u}(x)=0$ outside the interval $[c,d]$;
c. There exist real numbers a, b in $[c, d]$ such that
   (i). $\tilde{u}(x)$ monotonic increasing in $[c,a]$,
   (ii). $\tilde{u}(x)$ monotonic decreasing in $[b,d]$,
   (iii). $\tilde{u}(x)=1$, for $a \le x \le b$.

A more popular equivalent alternative definition of fuzzy number is as follows.

Definition **2.2.** A fuzzy number $\tilde{u}$ is a pair $\left(\underline{u}(r),\overline{u}(r)\right)$ of functions $\underline{u}(r),\overline{u}(r), 0 \le r \le 1$ which satisfy the following:

i. $\underline{u}(r)$ is a bounded left continuous non decreasing function over $[0,1]$;
ii. $\overline{u}(r)$ is a bounded left continuous non increasing function over $[0,1]$;
iii. $\underline{u}(r) \le \overline{u}(r), 0 \le r \le 1$.

The fuzzy number in Definition 2.2. is the fuzzy number in the parametric form.

A fuzzy number in triangular form $\tilde{u}=(a,\alpha,b)$ is the function

$$\mu_{\tilde{u}}(x)=\begin{cases}\dfrac{x-a}{\alpha-a}, & a \le x \le \alpha,\\ \dfrac{b-x}{b-\alpha}, & \alpha \le x \le b,\\ 0, & \text{otherwise.}\end{cases} \tag{2.1}$$

The above triangular fuzzy number can be written in the parametric form

$$\underline{u}(r)=(\alpha-a)r+a \quad \text{and}$$
$$\overline{u}(r)=b-(b-\alpha)r\,. \tag{2.2}$$

On the other hand, for arbitrary fuzzy number $\tilde{u}=\left(\underline{u}(r),\overline{u}(r)\right)$, the number

$$u_0=\tfrac{1}{2}\left(\underline{u}(1)+\overline{u}(1)\right) \tag{2.3}$$

is said to be the location index number of $\tilde{u}$, and two non-decreasing left continuous functions

$$\begin{aligned} u_*(r)&=u_0-\underline{u}(r)\\ u^*(r)&=\overline{u}(r)-u_0 \end{aligned} \tag{2.4}$$

are called the left fuzziness index function and the right fuzziness index function, respectively.

A triangular fuzzy number need not necessarily be of the form $\tilde{u}=(a,\alpha,b)$. In this paper, we write a fuzzy number in the form of $\tilde{u}=(a,\alpha,\beta)$, where a is the center, α is the left width and β is the right width. For example, if $\tilde{u}=(a,\alpha,b)=(1,5,7)$, then we have $\tilde{u}=(a,\alpha,\beta)$ as (5,4,2). For arbitrary fuzzy number $\tilde{u}=(a,\alpha,\beta)$, the membership function is of the form

$$\mu_{\tilde{u}}(x)=\begin{cases}1-\dfrac{a-x}{\alpha}, & a-\alpha \le x \le a,\\ 1-\dfrac{x-a}{\beta}, & a \le x \le a+\beta,\\ 0, & \text{otherwise.}\end{cases}$$

On the other hand, a parametric fuzzy number $\tilde{u}=\left[\underline{u}(r),\overline{u}(r)\right]$ can be represented as

$$\underline{u}(r)=a-(1-r)\alpha \ \text{ dan } \ \overline{u}(r)=a+(1-r)\beta\,.$$

Definition **2.3.** For arbitrary fuzzy numbers $\tilde{u}=\left(\underline{u}(r),\overline{u}(r)\right)$, $\tilde{v}=\left(\underline{v}(r),\overline{v}(r)\right)$ and real number k, we define

i. $\tilde{u}=\tilde{v}$ if and only if $\underline{u}(r)=\underline{v}(r)$ and $\overline{u}(r)=\overline{v}(r)$
ii. $\tilde{u}\oplus\tilde{v}=\left(\underline{u}(r)+\underline{v}(r),\overline{u}(r)+\overline{v}(r)\right)$
iii. $\tilde{u}\,\Theta\,\tilde{v}=\left(\underline{u}(r)-\overline{v}(r),\overline{u}(r)-\underline{v}(r)\right)$
iv. $\tilde{u}\otimes\tilde{v}=\left[\min\{\underline{u}(r)\underline{v}(r),\underline{u}(r)\overline{v}(r),\overline{u}(r)\underline{v}(r),\overline{u}(r)\overline{v}(r)\},\ \max\{\underline{u}(r)\underline{v}(r),\underline{u}(r)\overline{v}(r),\overline{u}(r)\underline{v}(r),\overline{u}(r)\overline{v}(r)\}\right]$

The product of two parametric fuzzy numbers is

- For $\tilde{u}\ge 0$ and $\tilde{v}\ge 0$, then $\tilde{u}\otimes\tilde{v}=\left(\underline{uv},\overline{uv}\right)$
- For $\tilde{u}\ge 0$ and $\tilde{v}\le 0$, then $\tilde{u}\otimes\tilde{v}=\left(\overline{u}\underline{v},\underline{u}\overline{v}\right)$

v. $k\tilde{u}=\begin{cases}\left(k\underline{u}(r),k\overline{u}(r)\right), & k\ge 0\\ \left(k\overline{u}(r),k\underline{u}(r)\right), & k<0\end{cases}$.

The definition of the product of these two fuzzy numbers give rise to the tedious work of checking the solution for DFFLS, since it is not easy to calculate the minimum and maximum. On the other hand, for the product of the parametric fuzzy numbers, the distributive law and the cancellation law do not hold.

Definition **2.4.** Fuzzy number $\tilde{u}$ is said to be positive (negative) denoted by $\tilde{u}>0$ $(\tilde{u}<0)$ if the membership function $\mu_{\tilde{u}}(x)=0, \forall x\le 0\ (\forall x\ge 0)$.

From definition 2.4 we note that $\tilde{u}=(a,\alpha,\beta)$ is positive (negative) if and only if $a - \alpha \ge 0$ ($a - \alpha \le 0$) and $\tilde{u}=(a,\alpha,\beta)$ is strictly positive (strictly negative) if and only if $a - \alpha > 0$ ($a - \alpha < 0$), zero triangular fuzzy number is $\tilde{0}=(0,0,0)$ and two fuzzy numbers $\tilde{u}=(a,\alpha,\beta)$ and $\tilde{v}=(b,\gamma,\delta)$ are equal if and only if $a=b, \alpha=\gamma$ dan $\beta=\delta$.

3. Algebra of Triangular Fuzzy Number

We give here the definition of the sum and the difference of two triangular fuzzy numbers. For the difference operation, we first transform the triangular fuzzy numbers into the parametric form, perform the difference operation in the parametric form and transform back the result into the triangular form, as have been discussed (Mashadi and Osman, 2009).

For $\tilde{u}=(a,\alpha,\beta)$ and $\tilde{v}=(b,\gamma,\delta)$, then the parametric forms are as follows:

$$\tilde{u}=(\underline{u}(r),\overline{u}(r))=[a-(1-r)\alpha,\, a+(1-r)\beta]$$

and

$$\tilde{v}=(\underline{v}(r),\overline{v}(r))=[b-(1-r)\gamma,\, b+(1-r)\delta].$$

Therefore

$$\begin{aligned}\tilde{w}_1=\tilde{u}\oplus\tilde{v}&=[a-(1-r)\alpha+b-(1-r)\gamma,\\ &\quad a+(1-r)\beta+b+(1-r)\delta]\\ &=[(a+b)-(1-r)(\alpha+\gamma),\\ &\quad (a+b)+(1-r)(\beta+\delta)]\end{aligned}$$

Transforming back into the triangular form, we have

$$\tilde{w}_2=\tilde{u}\oplus\tilde{v}=(a+b,\alpha+\gamma,\beta+\delta) \qquad (3.1)$$

which is is same as in Hosseini and Paripour (2009). Now we see that

$$\begin{aligned}\tilde{w}_3=\tilde{u}\Theta\tilde{v}&=[(a-(1-r)\alpha)-(b+(1-r)\delta),\\ &\quad (a+(1-r)\beta)-(b-(1-r)\gamma)]\\ &=[(a-b)-(1-r)(\alpha+\delta),\\ &\quad (a-b)+(1-r)(\beta+\gamma)]\end{aligned}$$

Transforming back into the triangular form, we have

$$\tilde{w}_4=\tilde{u}\Theta\tilde{v}=(a-b,\alpha+\delta,\beta+\gamma). \qquad (3.2)$$

On the other hand, Hosseini and Paripour, (2009) also define the inverse (minus) of a triangular fuzzy number as

$$-\tilde{u}=-(a,\alpha,\beta)=(-a,\beta,\alpha) \qquad (3.3)$$

and the scalar product as

$$\lambda\otimes\tilde{u}=\lambda\otimes(a,\alpha,\beta)=\begin{cases}(\lambda a,\lambda\alpha,\lambda\beta), & \lambda\geq 0;\\ (\lambda a,-\lambda\beta,-\lambda\alpha), & \lambda<0.\end{cases} \qquad (3.4)$$

From (3.1) we have that, for every triangular fuzzy number $\tilde{u}=(a,\alpha,\beta)$, there is a triangular fuzzy number $\tilde{v}=(-a,-\alpha,-\beta)$ such that $\tilde{u}\oplus\tilde{v}=\tilde{0}=(0,0,0)$. However from (3.2), we have that for every triangular fuzzy number $\tilde{u}=(a,\alpha,\beta)$, there is a triangular fuzzy number $\tilde{v}'=(a,-\beta,-\alpha)$ such that $\tilde{u}\Theta\tilde{v}'=\tilde{0}=(0,0,0)$. Now we can see that $\tilde{v}=(-a,-\alpha,-\beta)\ =-(a,-\beta,-\alpha)=-\tilde{v}'$ and hence the difference operation is of the form

$$\tilde{u}\,\Theta\tilde{v}'=(a,\alpha,\beta)\Theta(a,-\beta,-\alpha)=\tilde{0}=(0,0,0) \qquad (3.5)$$

Based on (3.5), we can perform the cancellation law on the equation $\tilde{A}\otimes\tilde{x}\oplus\tilde{b}=\tilde{C}\otimes\tilde{x}\oplus\tilde{d}$.

Next we perform the multiplication (product) of two triangular fuzzy numbers in various cases. In Dehghan and Hashemi (2006) only define the multiplication (product) of two triangular fuzzy numbers, i.e. if $\tilde{u}>0$ and $\tilde{v}>0$, then

$$\begin{aligned}\tilde{u}\otimes\tilde{v}&=(a,\alpha,\beta)\otimes(b,\gamma,\delta)\\ &=(ab,a\gamma+b\alpha,a\delta+b\beta),\end{aligned} \qquad (3.6a)$$

if $\tilde{u}<0$ and $\tilde{v}>0$, then

$$\begin{aligned}\tilde{u}\otimes\tilde{v}&=(a,\alpha,\beta)\otimes(b,\gamma,\delta)\\ &=(ab,b\alpha-a\delta,b\beta-a\gamma)\end{aligned} \qquad (3.6b)$$

This formulation will the be extended for various cases of $\tilde{u}$ and $\tilde{v}$.

On the other hand is Rouhvarpar and Allahviranloo, (2007) showed that the multiplication(product) of two positive fuzzy numbers is as follows:

***Theorem* 3.1**. If $\tilde{u}=(\underline{u}(r),\overline{u}(r))$ and $\tilde{v}=(\underline{v}(r),\overline{v}(r))$ are two positive fuzzy numbers, then $\tilde{w}=\tilde{u}\otimes\tilde{v}=(\underline{w}(r),\overline{w}(r))$, where

$$\underline{w}(r)=\underline{u}(r)\underline{v}(1)+\underline{u}(1)\underline{v}(r)-\underline{u}(1)\underline{v}(1), \qquad (3.7)$$

for every $r\in[0,1]$, is a positive fuzzy number.

Based on Theorem 3.1, for any two fuzzy numbers $\tilde{u}=(\underline{u}(r),\overline{u}(r))$ and $\tilde{v}=(\underline{v}(r),\overline{v}(r))$, we have the following:

i. If $\tilde{u}$ is positive and $\tilde{v}$ is negative, then $\tilde{w}=-(\tilde{u}\otimes(-\tilde{v}))$ is negative;

ii. If $\tilde{u}$ is negative and $\tilde{v}$ is positive, then $\tilde{w}=-((-\tilde{u}\otimes\tilde{v}))$ is negative;

iii. If $\tilde{u}$ negative and $\tilde{v}$ is negative, then $\tilde{w}=(-(\tilde{u})\otimes(-\tilde{v}))$ is positive.

Now based on Theorem 3.1 and the above multiplication (i – iii), then the multiplication (product) of any two fuzzy numbers $\tilde{u}=(\underline{u}(r),\overline{u}(r))$ and $\tilde{v}=(\underline{v}(r),\overline{v}(r))$ can be shown that for every $r\in[0,1]$:

iv. If $\tilde{u}$ is positive and $\tilde{v}$ is negative, then

$$\begin{cases}\underline{w}(r)=\overline{u}(r)\underline{v}(1)+\overline{u}(1)\underline{v}(r)-\overline{u}(1)\underline{v}(1)\\ \overline{w}(r)=\underline{u}(r)\overline{v}(1)+\underline{u}(1)\overline{v}(r)-\underline{u}(1)\overline{v}(1)\end{cases} \qquad (3.8)$$

v. If $\tilde{u}$ is negative and $\tilde{v}$ is positive, then

$$\begin{cases}\underline{w}(r)=\underline{u}(r)\overline{v}(1)+\underline{u}(1)\overline{v}(r)-\underline{u}(1)\overline{v}(1)\\ \overline{w}(r)=\overline{u}(r)\underline{v}(1)+\overline{u}(1)\underline{v}(r)-\overline{u}(1)\underline{v}(1)\end{cases} \qquad (3.9)$$

vi. If $\tilde{u}$ is negative and $\tilde{v}$ is negative, then

$$\begin{cases}\underline{w}(r)=\overline{u}(r)\overline{v}(1)+\overline{u}(1)\overline{v}(r)-\overline{u}(1)\overline{v}(1)\\ \overline{w}(r)=\underline{u}(r)\underline{v}(1)+\underline{u}(1)\underline{v}(r)-\underline{u}(1)\underline{v}(1)\end{cases} \qquad (3.10)$$

Based on the above condition, we now extend to the cross product for arbitrary two triangular fuzzy numbers $\tilde{u}=(a,\alpha,\beta)$ dan $\tilde{v}=(b,\gamma,\delta)$ with the addition and scalar product as in (3.1), (3.3) and (3.4) while the cross product for positive $\tilde{u}$ and positive $\tilde{v}$ as in (3.6a). Note that this can be seen as follows:

Let $\tilde{u}=[\underline{u}(r),\overline{u}(r)]=[(a-(1-r)\alpha),(a+(1-r)\beta)]$,

$\tilde{v}=[\underline{v}(r),\overline{v}(r)]=[(b-(1-r)\gamma),(b+(1-r)\delta)]$.

Then the center of $\tilde{w}$ is $w_0=ab$. From (3.7) we have,

$$\begin{aligned}\tilde{w}=[\underline{w}(r),\overline{w}(r)]&=[(a-(1-r)\alpha)b+a(b-(1-r)\gamma)-ab,\\&\quad(a+(1-r)\beta)b+a(b+(1-r)\delta)-ab]\\&=[ab-(1-r)(a\gamma+b\alpha),ab+(1-r)(a\delta+b\beta)]\end{aligned}\tag{3.11}$$

If we let $\tilde{w}=(c,\xi,\psi)$, then the parametric fuzzy number is of the form

$$\tilde{w}=[c-(1-r)\xi,c+(1-r)\psi]\tag{3.12}$$

From equation (2.2), (3.11) and (3.12), we have

$$\xi=a\gamma+b\alpha \quad\text{and}\quad \psi=a\delta+b\beta$$

Hence $\tilde{w}=\tilde{u}\otimes\tilde{v}$ in a triangular fuzzy number for positive $\tilde{u}$ and positive $\tilde{v}$ can be written as

$$\tilde{w}=\tilde{u}\otimes\tilde{v}=(ab,a\gamma+b\alpha,a\delta+b\beta).\tag{3.13}$$

Now for positive $\tilde{u}$ and negative $\tilde{v}$, the cross product is

$$\begin{aligned}\tilde{w}=[\underline{w}(r),\overline{w}(r)]&=[(a+(1-r)\beta)b+a(b-(1-r)\gamma)-ab,\\&\quad(a-(1-r)\alpha)b+a(b+(1-r)\delta)-ab]\\&=[ab-(1-r)(a\gamma-b\beta),ab+(1-r)(a\delta-b\alpha)]\end{aligned}\tag{3.14}$$

Since the center does not change, from (2.2) and (3.14), we have

$$\tilde{w}=\tilde{u}\otimes\tilde{v}=(ab,a\gamma-b\beta,a\delta-b\alpha).\tag{3.15}$$

Again for negative $\tilde{u}$ and positive, the cross product is

$$\begin{aligned}\tilde{w}=[\underline{w}(r),\overline{w}(r)]&=[(a-(1-r)\alpha)b+a(b+(1-r)\delta)-ab,\\&\quad(a+(1-r)\beta)b+a(b-(1-r)\gamma)-ab]\\&=[ab-(1-r)(-a\delta+b\alpha),ab+(1-r)(-a\gamma+b\beta)]\end{aligned}\tag{3.16}$$

Since the center does not change, from (2.2) and (3.16), we have

$$\tilde{w}=\tilde{u}\otimes\tilde{v}=(ab,-a\delta+b\gamma,-a\gamma+b\beta)\tag{3.17}$$

Finally for negative $\tilde{u}$ and negative $\tilde{v}$, the cross product is

$$\begin{aligned}\tilde{w}=[\underline{w}(r),\overline{w}(r)]&=[(a+(1-r)\beta)b+a(b+(1-r)\delta)-ab,\\&\quad(a-(1-r)\alpha)b+a(b-(1-r)\gamma)-ab]\\&=[ab+(1-r)(a\delta+b\beta),ab-(1-r)(a\gamma+b\alpha)]\end{aligned}\tag{3.18}$$

Since the center do not changed, from (2.2) and (3.18) , we have

$$\tilde{w}=\tilde{u}\otimes\tilde{v}=(ab,-(a\delta+b\beta),-(a\gamma+b\alpha)).\tag{3.19}$$

4. Solving DFFLS $\tilde{A}\otimes\tilde{x}\oplus\tilde{b}=\tilde{C}\otimes\tilde{x}\oplus\tilde{d}$

Let the dual fully fuzzy linear system (DFFLS) be as follows:

$$\begin{cases}\tilde{a}_{11}\tilde{x}_1\oplus\tilde{a}_{12}\tilde{x}_2\oplus\cdots\oplus\tilde{a}_{1n}\tilde{x}_n\oplus\tilde{b}_1=\tilde{c}_{11}\tilde{x}_1\oplus\tilde{c}_{12}\tilde{x}_2\oplus\cdots\oplus\tilde{c}_{1n}\tilde{x}_n\oplus\tilde{d}_1\\ \tilde{a}_{21}\tilde{x}_1\oplus\tilde{a}_{22}\tilde{x}_2\oplus\cdots\oplus\tilde{a}_{2n}\tilde{x}_n\oplus\tilde{b}_2=\tilde{c}_{21}\tilde{x}_1\oplus\tilde{c}_{22}\tilde{x}_2\oplus\cdots\oplus\tilde{c}_{2n}\tilde{x}_n\oplus\tilde{d}_2\\ \vdots\qquad\vdots\qquad\ddots\qquad\vdots\qquad\vdots\qquad\vdots\qquad\vdots\qquad\ddots\qquad\vdots\qquad\vdots\\ \tilde{a}_{n1}\tilde{x}_1\oplus\tilde{a}_{n2}\tilde{x}_2\oplus\cdots\oplus\tilde{a}_{nn}\tilde{x}_n\oplus\tilde{b}_n=\tilde{c}_{n1}\tilde{x}_1\oplus\tilde{c}_{n2}\tilde{x}_2\oplus\cdots\oplus\tilde{c}_{nn}\tilde{x}_n\oplus\tilde{d}_n\end{cases}\tag{4.1}$$

We denote (4.1) in the matrix form as $\tilde{A}\otimes\tilde{x}\oplus\tilde{b}=\tilde{C}\otimes\tilde{x}\oplus\tilde{d}$ where $\tilde{A}$ and $\tilde{C}$ are matrices with fuzzy numbers as its elements, while $\tilde{b}$ and $\tilde{d}$ are fuzzy vectors. Since the basic property of a triangular fuzzy number is $\tilde{u}\Theta\tilde{u}\neq\tilde{0}$, then equation $\tilde{A}\otimes\tilde{x}\oplus\tilde{b}=\tilde{C}\otimes\tilde{x}\oplus\tilde{d}$ is not equivalent to $(\tilde{A}\Theta\tilde{C})\otimes\tilde{x}=\tilde{d}\Theta\tilde{b}$.

Theorem 4.1. [4] Let A be an n × n matrix with all non-zero leading principal minors. Then A has a unique factorizations $A = LU$, where L is unit lower triangular and U is upper triangular.

Assume that $\tilde{A}=(A,M,N)$ and $\tilde{C}=(C,K,L)$, where A and C are full rank crisp matrices. Then if we let

$$(L_1,0,0)\otimes(U_1,U_2,U_3)=(A,M,N)\tag{4.2}$$

and

$$(L_1^\circ,0,0)\otimes(U_1^\circ,U_2^\circ,U_3^\circ)=(C,K,L)\tag{4.3}$$

then we have

$$\begin{cases}L_1U_1=A & \Rightarrow U_1=L_1^{-1}A\\ L_1U_2=M & \Rightarrow U_2=L_1^{-1}M\\ L_1U_3=N & \Rightarrow U_3=L_1^{-1}N\end{cases}\tag{4.4}$$

and

$$\begin{cases}L_1^\circ U_1^\circ=C & \Rightarrow U_1^\circ=(L_1^\circ)^{-1}C\\ L_1^\circ U_2^\circ=K & \Rightarrow U_2^\circ=(L_1^\circ)^{-1}K\\ L_1^\circ U_3^\circ=L & \Rightarrow U_3^\circ=(L_1^\circ)^{-1}L\end{cases}\tag{4.5}$$

Consider the dual fully fuzzy linear system. We are going to construct solutions for $\tilde{A}\otimes\tilde{x}\oplus\tilde{b}=\tilde{C}\otimes\tilde{x}\oplus\tilde{d}$, where $\tilde{A}=(A,M,N)$ and $\tilde{C}=(C,K,L)$, $\tilde{x}=(x,y,z)$, $\tilde{b}=(b,g,h)$ and $\tilde{d}=(d,s,t)$, that is

$$\begin{aligned}&(A,M,N)\otimes(x,y,z)\oplus(b,g,h)\\&=(C,K,L)\otimes(x,y,z)\oplus(d,s,t)\end{aligned}.$$

Equations (4.2) and (4.3) implies that

$$\left(L_1U_1, L_1U_2, L_1U_3\right)\otimes(x,y,z)\oplus(b,g,h) = \left(L_1^{\circ}U_1^{\circ}, L_1^{\circ}U_2^{\circ}, L_1^{\circ}U_3^{\circ}\right)\otimes(x,y,z)\oplus(d,s,t).$$

Therefore, by using (3.6a) we have

$$\left(L_1U_1x, L_1U_2x + L_1U_1y, L_1U_3x + L_1U_1z\right) \oplus(b,g,h) = \left(L_1^{\circ}U_1^{\circ}x, L_1^{\circ}U_2^{\circ}x + L_1^{\circ}U_1^{\circ}y, L_1^{\circ}U_3^{\circ}x + + L_1^{\circ}U_1^{\circ}z\right)\oplus(d,s,t).$$

Next by (3.5) we have

$$\left(L_1U_1x, L_1U_2x + L_1U_1y, L_1U_3x + L_1U_1z\right) = \left(L_1^{\circ}U_1^{\circ}x, L_1^{\circ}U_2^{\circ}x + L_1^{\circ}U_1^{\circ}y, L_1^{\circ}U_3^{\circ}x + + L_1^{\circ}U_1^{\circ}z\right)\oplus(d,s,t)\Theta(b,-h,-g).$$

Let $\tilde{f} = (f,p,q) = (d,s,t)\Theta(b,-h,-g)$, then we have

$$\left(L_1U_1x, L_1U_2x + L_1U_1y, L_1U_3x + L_1U_1z\right) = \left(L_1^{\circ}U_1^{\circ}x, L_1^{\circ}U_2^{\circ}x + L_1^{\circ}U_1^{\circ}y, L_1^{\circ}U_3^{\circ}x + L_1^{\circ}U_1^{\circ}z\right) \oplus(f,p,q)$$

$$\left(L_1U_1x, L_1U_2x + L_1U_1y, L_1U_3x + L_1U_1z\right) = \left(L_1^{\circ}U_1^{\circ}x + f, L_1^{\circ}U_2^{\circ}x + L_1^{\circ}U_1^{\circ}y + + p, L_1^{\circ}U_3^{\circ}x + L_1^{\circ}U_1^{\circ}z + q\right) \qquad (4.6)$$

By equality in fuzzy number, equations (4.6) can be rewrite as follows:

$$\begin{cases} L_1U_1x = L_1^{\circ}U_1^{\circ}x + f \\ L_1U_2x + L_1U_1y = L_1^{\circ}U_2^{\circ}x + L_1^{\circ}U_1^{\circ}y + p \\ L_1U_3x + L_1U_1z = L_1^{\circ}U_3^{\circ}x + L_1^{\circ}U_1^{\circ}z + q. \end{cases} \qquad (4.7)$$

On the equations (4.7) $L_1, L_1^{\circ}, U_1, U_2, U_2, U_3, U_1^{\circ}, U_2^{\circ}$ and U_3° are usual lower and upper triangular crisp matrices, so we can now do some algebraic operations on these matrices (4.7) as follows

$$\begin{cases} \left(L_1U_1 - L_1^{\circ}U_1^{\circ}\right)x = f \\ \left(L_1U_2 - L_1^{\circ}U_2^{\circ}\right)x + \left(L_1U_1 - L_1^{\circ}U_1^{\circ}\right)y = p \\ \left(L_1U_3 - L_1^{\circ}U_3^{\circ}\right)x + \left(L_1U_1z - L_1^{\circ}U_1^{\circ}\right)z = q, \end{cases}$$

and therefore

$$\begin{cases} x = \left(L_1U_1 - L_1^{\circ}U_1^{\circ}\right)^{-1} f \\ y = \left(L_1U_1 - L_1^{\circ}U_1^{\circ}\right)^{-1}\left(p - \left(L_1U_2 - L_1^{\circ}U_2^{\circ}\right)x\right) \\ z = \left(L_1U_1z - L_1^{\circ}U_1^{\circ}\right)^{-1}\left(q - \left(L_1U_3 - L_1^{\circ}U_3^{\circ}\right)x\right). \end{cases} \qquad (4.8)$$

Computational Example: Solve the following dual fully fuzzy linear system:

$$\begin{cases} (2,1,1)\tilde{x}_1 \oplus (6,2,3)\tilde{x}_2 \oplus (3,2,1) = (1,1,2)\tilde{x}_1 \oplus (6,1,2)\tilde{x}_2 \oplus (5,4,4) \\ (6,3,2)\tilde{x}_1 \oplus (5,2,1)\tilde{x}_2 \oplus (3,1,1) = (4,3,2)\tilde{x}_1 \oplus (3,1,1)\tilde{x}_2 \oplus (9,5,5) \end{cases} \qquad (4.10)$$

Solution : First we obtain *LU*-decompositions for matrices *A* and *C* as follows :

$$A = \begin{bmatrix} 2 & 6 \\ 6 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 2 & 6 \\ 0 & -13 \end{bmatrix} = L_1U_1$$

and

$$C = \begin{bmatrix} 1 & 6 \\ 4 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix}\begin{bmatrix} 1 & 6 \\ 0 & -21 \end{bmatrix} = L_1^{\circ}U_1^{\circ}$$

So we can obtain matrices U_2, U_3 and U_2°, U_3° as follows

$$U_2 = L_1^{-1}M = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & -4 \end{bmatrix}$$

$$U_3 = L_1^{-1}N = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ -1 & -8 \end{bmatrix}$$

$$U_2^{\circ} = \left(L_1^{\circ}\right)^{-1}K = \begin{bmatrix} 1 & 0 \\ -4 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & -3 \end{bmatrix}$$

$$U_3^{\circ} = \left(L_1^{\circ}\right)^{-1}L = \begin{bmatrix} 1 & 0 \\ -4 & 1 \end{bmatrix}\begin{bmatrix} 2 & 2 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ -6 & -7 \end{bmatrix}$$

Next we obtain

$$\tilde{f}_1 = (f_1, p_1, q_1) = (d_1, s_1, t_1)\Theta(b_1, -h_1, -g_1) = (5,4,4)\Theta(3,-1,-2) = (2,2,3)$$

and

$$\tilde{f}_2 = (f_2, p_2, q_2) = (d_2, s_2, t_2)\Theta(b_2, -h_2, -g_2) = (9,5,5)\Theta(3,-1,-1) = (6,4,4)$$

hence $\tilde{f} = (f,p,q) = \begin{bmatrix} (2,2,3) \\ (6,4,4) \end{bmatrix}$.

Therefore, by equations (4.10) conclude that

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \left(L_1U_1 - L_1^{\circ}U_1^{\circ}\right)^{-1} f = \left(\begin{bmatrix} 2 & 6 \\ 6 & 5 \end{bmatrix} - \begin{bmatrix} 1 & 6 \\ 4 & 3 \end{bmatrix}\right)^{-1}\begin{bmatrix} 2 \\ 6 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}\left(L_1U_1 - L_1^{\circ}U_1^{\circ}\right)^{-1}\left(p - \left(L_1U_2 - L_1^{\circ}U_2^{\circ}\right)x\right)$$

$$= \begin{bmatrix} 1 & 0 \\ -1 & 0.5 \end{bmatrix}\left(\begin{bmatrix} 2 \\ 4 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right) =$$

$$= \begin{bmatrix} 1 & 0 \\ -1 & 0.5 \end{bmatrix}\begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0.5 \end{bmatrix}$$

$$z = \left(L_1U_1 - L_1^{\circ}U_1^{\circ}\right)^{-1}\left(q - \left(L_1U_3 - L_1^{\circ}U_3^{\circ}\right)x\right)$$

$$= \begin{bmatrix} 1 & 0 \\ -1 & 0.5 \end{bmatrix}\left(\begin{bmatrix} 3 \\ 4 \end{bmatrix} - \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right)$$

$$= \begin{bmatrix} 1 & 0 \\ -1 & 0.5 \end{bmatrix}\begin{bmatrix} 4 \\ 4 \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \end{bmatrix},$$

therefore $z_1 = 4$ dan $z_2 = -2$.

Hence we have the solution $\tilde{x}_1 = (x_1, y_1, z_1) = (2,1,4)$ and $\tilde{x}_2 = (x_2, y_2, z_2) = (1,0.5,-2)$.

To check the validity of the solution, we substitute back $\tilde{x}_1$ and $\tilde{x}_2$ into the original equation (4.8) using the multiplication rule as in (3.13), (3.15), (3.17) and (3.19), while the addition and subtraction rule as in (3.1) and (3.2). For the above solution $\tilde{x}_1$ and $\tilde{x}_2$, we have :

The left-hand side of the first equation as:

$(2,1,1)\otimes(2,1,4) \oplus (6,2,3)\otimes(1,0.5,-2) \oplus (3,2,1) =$
$(4,4,10) \oplus (6,5,-9) \oplus (3,2,1) = (13,11,2)$

The right-hand side of the first equation as:

$(1,1,2)\otimes(2,1,4) \oplus (6,1,2)\ )\otimes(1,0.5,-2) \oplus (5,4,4) =$
$(2,3,8) \oplus (6,4,-10) \oplus (5,4,4) = (13,11,2)$

The left-hand side of the second equation as:

$(6,3,2)\otimes(2,1,4) \oplus (5,2,1)\otimes(1,0.5,-2) \oplus (3,1,1) =$
$(12,12,28) \oplus (5,4.5,-9) \oplus 3,1,1) = (20,17.5,20)$

The right-hand side of the second equation as:

$(4,3,2)\otimes(2,1,4) \oplus (3,1,1)\otimes(1,0.5,-2) \oplus (9,5,5) =$
$(8,10,20) \oplus (3,2.5,-5) \oplus (9,5,5) = (20,17.5,20)$

Hence we see that the solution is true.

5. Conclusion

In this paper we propose a new method for solving dual fully fuzzy linear systems $\tilde{A} \otimes \tilde{x} \oplus \tilde{b} = \tilde{C} \otimes \tilde{x} \oplus \tilde{d}$, where $\tilde{A}$ and $\tilde{C}$ are the coefficient matrices of triangular fuzzy numbers while $\tilde{b}$ and $\tilde{d}$ are fuzzy vectors. We further validate the solution to be true by substituting in the original equation.

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