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SOME RESULT ON EXCIRCLE OF QUADRILATERAL

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Abstract

Any quadrilateral not necessarily have excircle, in this paper will discuss necessary and sufficient condition that any quadrilateral having excircle. It also will set the various lengths of the sides are formed from the construction result of excircle. Besides that we can also establish some other excircle and also be specified the length of radii and the relationship of the radii with the presence of excircle.

1. Background

In a triangle, can always be constructed incircle and excircle [3, 13], while in any quadrilateral, may not necessarily be formed incenter and the excenter. Several authors discuss the incircle of a quadrilateral is [4-7, 9, 11], the author only discusses about the comparative of radii, side and diagonal of the quadrilateral.

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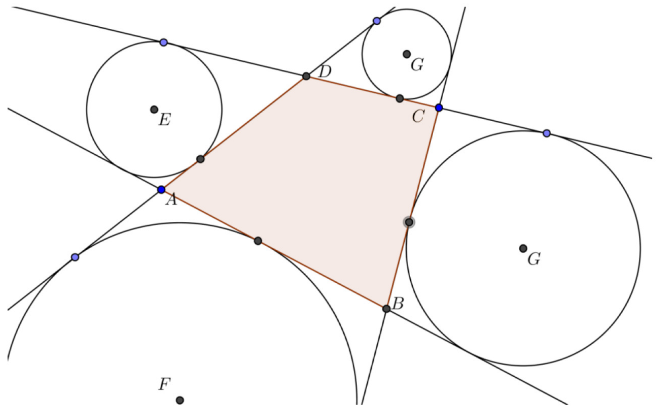

Figure 1.1

At [1], the author discusses the excircle as in Figure 1.1, which if we look, then it is not an excircle, because each circle just touches one side and extension of the other two sides. Furthermore, [8, 9] to construct an excircle (as Figure 1.2), which gives the requirements that the quadrilateral *ABCD* with sides $AB = a$, $BC = b$, $CD = c$ and $DA = d$, then *ABCD* is tangential quadrilateral if $a + c = b + d$.

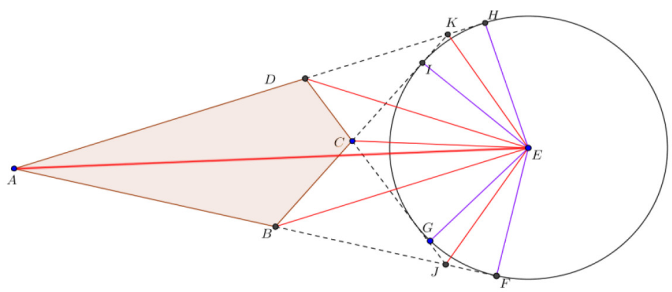

Figure 1.2

However [8] also explained that if *ABCD* convex quadrilateral where opposite sides *AB* and *CD* intersect at J, and the sides *AD* and *BC* intersect at *K* (see, Figure 1.3), then *ABCD* is a tangential quadrilateral if and only if either of

$$BJ + BK = DJ + DK,$$

$$AJ - AK = CJ - CK.$$

In addition to the [9] also indicated that if the *ABCD* convex quadrilateral with sides a, b, c, d has an excircle if and only if $|a - c| = |b - d|$ as well as to construct various subtriangles so it obtained four pieces circumradii order to obtain the various relationships between the fourth circumradii. If we look at Figure 1.2 above, of course, the most important first is to prove that the bisector $\angle A$, external bisector $\angle B$ and $\angle D$ and opposite bisector $\angle C$ are concurrence which then needs to be discussed how the radii of the excircle as well as a variety of side length resulting from the excircle constructing.

2. Theoretical Basis

In addition to calculating the radii of the circle tangent, can also be calculated distance of the center point to the one point of the outer angles of the triangle (external bisector). Josefsson in [7] lowers the formula in Lemma 2.1.

Lemma 2.1. *If O is the center of the incircle, then* ΔABC *will apply*

$$\frac{OB^2}{ac} = \frac{s-c}{s-a},$$

with is a semiperimeter.

Proof. See [10].

The constructed excircle on a convex quadrilateral produce concurrency of six bisector angle. To prove concurrency of six bisector angle, can be done in the following manner. Note Figure 2.1.

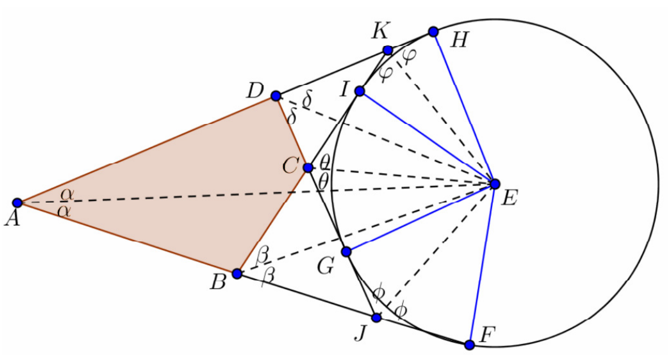

Figure 2.1

Note ΔABK in Figure 2.1. Since of the circle centered at *E* offending side of *BK* in point *I*, the extension of *AK* at the point *H* and the extension of *AB* at point *F*, the circle is excircle of ΔABK. By making a line bisector angle of each $\angle A$, $\angle KBF$, and $\angle BKH$, then the third angle of bisector concurrence at the point E. Thus, all three lines of bisector are AE, BE, and EK concurrence at point E.

Note ΔADJ. Since the circle centered at E offend the extension side of AD at point H, AJ extension at point F and DJ at point G, the circle is excircle from ΔADJ. By creating a line bisector angle of each $\angle A$, $\angle JDK$, and $\angle DJF$ third-line of the bisector angle, namely AE, DE, and EJ concurrence at point E.

Points I and G is a point of tangency of circle. By connecting the points C and E are formed two triangles, namely ΔCIE and ΔCEG, so $\angle ICE = \angle GCE$ meaning that CE is bisector line of $\angle KCJ$. Thus proven AE, BE, CE, DE, EJ and EK concurrence.

Theorem 2.2. *An outer circle tangent quadrilateral with sides* a, b, c *and* d *has the length of radii*

$$\rho = \frac{L\square ABCD}{a - c}$$

$$= \frac{L\square ABCD}{d - b}.$$

Proof. See [10].

Theorem 2.3. *Suppose the* $\square ABCD$ *with* $AB = a$, $BC = b$, $CD = c$ *and* $AD = d$ *and also has a tangential excircle. Then*

$$L\square ABCD^2 = abcd(\sin \gamma)^2,$$

where 2γ *is the number of opposite angles.*

Proof. Consider Figure 2.2. Pull the line BD, so that there are two triangles that ΔABD and ΔBDC at ΔABD applies: $BD^2 = a^2 + d^2 - 2ad \cos \angle A$. And on the ΔBDC the same thing is applicable that $BD^2 = b^2 + c^2 - 2bc \cos \angle C$.

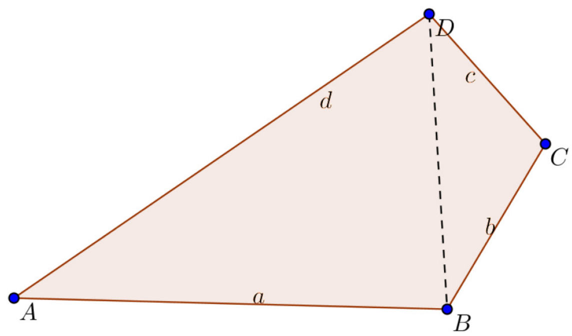

Figure 2.2

Thus obtained

$$a^2 + d^2 - 2ad\cos\angle A = b^2 + c^2 - 2bc\cos\angle C,$$

$$(a^2 + d^2 - b^2 - c^2)^2 = 4(ad\cos\angle A - bc\cos\angle C)^2. \qquad (2.1)$$

Additionally, $L\square ABCD$ can be written as

$$L\square ABCD = L\square ABD + L\square BCD$$

$$= \frac{1}{2}ad\sin\angle A + \frac{1}{2}bc\sin\angle C,$$

$$16L\square ABCD^2 = 4(ad\sin\angle A + bc\sin\angle C)^2 \qquad (2.2)$$

add the equation (3.15) with (2.2) is obtained

$$(a^2 + d^2 - b^2 - c^2)^2 + 16L\square ABCD^2$$

$$= 4(ad\cos\angle A - bc\cos\angle C)^2 + 4(ad\sin\angle A + bc\sin\angle C)^2$$

$$= 4(a^2d^2 + b^2c^2) - 16abcd(\cos\gamma)^2 + 8abcd$$

$$= (2ad + 2bc)^2 - (a^2 + d^2 - b^2 - c^2)^2 - 16abcd(\cos\gamma)^2$$

$$= [(2ad + 2bc) - (a^2 + d^2 - b^2 - c^2)^2][(2ad + 2bc) + (a^2 + d^2 - b^2 - c^2)]$$

$$- 16abcd(\cos\gamma)^2$$

$$= 16(s-a)(s-b)(s-c)(s-d) - 16abcd(\cos\gamma)^2$$

since $a + b = c + d = s$ so

$$16L\square ABCD^2 = 16abcd - 16abcd(\cos\gamma)^2,$$

$$L\square ABCD^2 = abcd(\sin\gamma)^2. \qquad \blacksquare$$

3. Sides and Radii

3.1. Sides length

The constructed of excircle on the quadrilateral produce various sides. For more details, please see Figure 3.1.

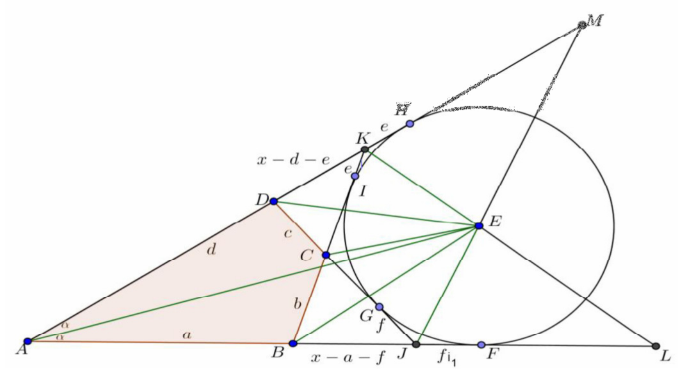

Figure 3.1

In Figure 3.1, there exists $\square ABCD$ with $\angle A = 2\alpha$. Suppose the length of $HK = e$ and $FJ = f$. Since IK and HK are the tangents of the point K, $IK = HK = e$. Furthermore, FJ and GJ are also tangents from the point F so that $GJ = FJ = f$. Let $AH = x$. Then $DK = x - d - e$.

Furthermore, since of AH and AF are tangents from point A, $AH = AF$, so $BJ = x - a - f$. Since the DG and DH are also tangents from point D, $DG = DH$, so that $DG = x - d$, which produces $CG = x - c - d$.

Furthermore, since $CI = CG = x - c - d$, the circle centered at point E is excircle from four triangles, namely ΔABK, ΔADJ, ΔBCJ and ΔCDK. Thus the *semiperimeter* (s) of each triangle is

$$s_1 = s\Delta ABK$$

$$= \frac{a + b + x - c - d + e + x - d - e + d}{2},$$

$$s_1 = \frac{a + b - c - d + 2x}{2}.$$

Since $AH = AF = s\Delta ABK = \Delta ADJ = x$, call $s_2 = s\Delta ADJ = x$, and then we have

$$s_3 = s\Delta BCK$$

$$= x - a, \tag{3.1}$$

$$s_4 = s\Delta CDK$$

$$= x - d. \tag{3.2}$$

By using Lemma 2.1 in ΔABK and equation (3.1),

$$\frac{BE^2}{AB \times BK} = \frac{x - AB}{x - BK},$$

$$\frac{BE^2}{a(b + x - c - d + e)} = \frac{x - a}{x - (b + x - c - d + e)},$$

$$BE^2 = \frac{x - a}{c + d - b - e} a(b + x - c - d + e) \tag{3.3}$$

and in a similar way would be obtained

$$EK^2 = \frac{e}{c + d - b - e}(x - e)(b + x - c - d + e), \tag{3.4}$$

$$EJ^2 = \frac{f}{d - f}(x - f)(x - d + f), \tag{3.5}$$

$$DE^2 = \frac{(x - d)}{d - f} d(x - d + f), \tag{3.6}$$

$$JE^2 = \frac{f}{c + d - f - a}(x - a - f)(x - c - d + f), \tag{3.7}$$

$$CE^2 = \frac{(x-a-b)}{(c+d-f-a)} b(x-c-d+f), \tag{3.8}$$

$$EK^2 = \frac{e}{c-e}(x-d-e)(x-c-d+e). \tag{3.9}$$

By substituting the equation (3.4) to (3.8) is obtained

$$e(x-e)(b+x-c-d+e)(c-e)$$

$$= e(x-d-e)(x-c-d+e)(c+d-b-e),$$

$$e = \frac{d(x-c)^2 - b(x-d)^2 - 2xd^2 - bcd + 2cd^2 + d^3}{(d^2+cd-dx-bx)}. \tag{3.10}$$

Furthermore to find the value of f, substitution of equation (3.5) to the equation (3.7) is obtained

$$f(x-f)(x-d+f)(c+d-f-a) = f(x-a-f)(x-c-d+f)(d-f),$$

$$f = \frac{-cx^2 + ax^2 - 2adx + acd + ad^2}{ad + ac - cx - ax}. \tag{3.11}$$

Note the ΔAEH. Since $\angle HAE = \alpha,$ it is obtained

$$\frac{AH}{\sin(90^\circ - \alpha)} = \frac{EH}{\sin\alpha},$$

$$AH = \frac{EH}{\sin\alpha}\cos\alpha.$$

In a similar way also be obtained

$$e = \frac{cd[-2L\square ABCD \cot\alpha + (a-c)(c-b-2d)]}{L\square ABCD \cot\alpha(-b-d) + ad - bcd}. \tag{3.12}$$

By substituting the equation (3.8) to the equation (3.7) is obtained

$$f = \frac{acd(a-c-d)}{(L\square ABCD \cot\alpha(-a-c) + (a-c)(ad+ac))} \tag{3.13}$$

from x, e, and f and $DH = x - d$, then

$$DH = \frac{L\Box ABCD}{a-c}\cot\alpha - d,$$

$$CI = \frac{L\Box ABCD}{a-c}\cot\alpha - a - b.$$

Furthermore, by using the principle of Pythagoras obtained

$$AE^2 = \left(\frac{L\Box ABCD}{a-c}\cot\alpha\right)^2 + \left(\frac{L\Box ABCD}{a-c}\right)^2$$

$$= \left(\frac{L\Box ABCD}{a-c}\right)^2 ((\cot\alpha)^2 + 1),$$

$$BE^2 = \left(\frac{L\Box ABCD}{a-c}\cot\alpha - a\right)^2 + \left(\frac{L\Box ABCD}{a-c}\right)^2$$

$$= \frac{[L\Box ABCD\cot\alpha - a(a-c)]^2 + L\Box ABCD^2}{(a-c)^2},$$

$$CE^2 = \left(\frac{L\Box ABCD}{a-c}\cot\alpha - a - b\right)^2 + \left(\frac{L\Box ABCD}{a-c}\right)^2,$$

$$DE^2 = \left(\frac{L\Box ABCD}{a-c}\cot\alpha - d\right)^2 + \left(\frac{L\Box ABCD}{a-c}\right)^2.$$

3.2. Radii of the other excircle

Note Figure 3.2. If incircle formed on ΔBCJ centered in O_b and ΔCDK based in O_c, the circle is also an excircle. Thus the length of radii of excircle on $\Box ABCD$ that offensive side of b and c can be solved by using excircle theorem for the triangle.

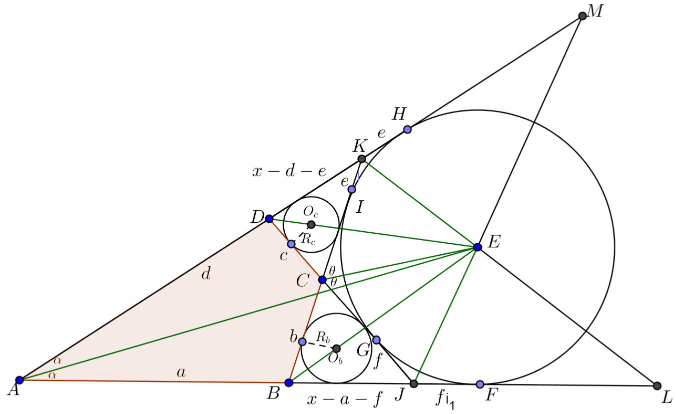

Figure 3.2

Note ΔBCJ. Suppose $\angle BCD = 2\theta$. Then $\angle BCJ = 180^\circ - 2\theta$. So the length of the radii is symbolized by R_b is

$$R_b = (s\Delta BCJ - BJ)\tan\frac{1}{2}(180^\circ - 2\theta)$$

$$= (BJ + JF - BJ)\tan(90^\circ - \theta)$$

$$= f\cot\theta,$$

$$R_b = \frac{acd(a-c-d)}{L\square ABCD\cot\alpha(-a-c)+(a-c)(ad+ac)}\cot\theta.$$

In a similar way would be obtained

$$R_c = \frac{cd[-2L\square ABCD\cot\alpha + (a-c)(c-b-2d)]}{L\square ABCD\cot\alpha(-b-d)+ad-bcd}\cot\theta.$$

3.3. Relationship of radii of the other excircle

The following are given characteristics of quadrilateral that has circle tangent a second shape that associated with the circle tangent first form.

Theorem 3.4. *Given a* $\square ABCD$ *has excircle if and only if* $R_aR_b = R_cR_d$.

Proof. ($\Rightarrow$) Suppose a $\square ABCD$ has excircle with long of radii R_a, R_b, R_c and R_d, and also has a excircle in front of the point of C. Will be shown that $R_aR_b = R_cR_d$.

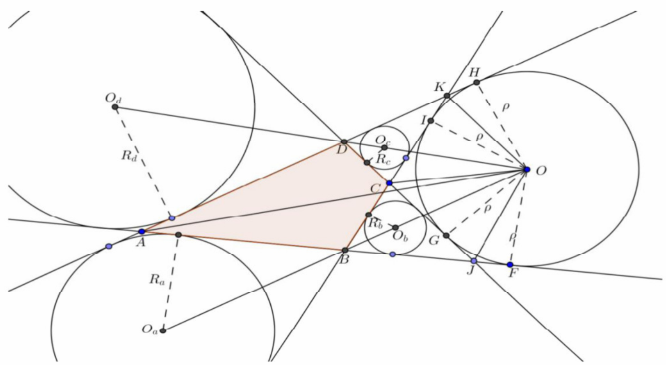

Figure 3.3

Note Figure 3.3. a $\square ABCD$ the circle is a circle that is centered on O_a offensive in side a at the point of L, O_b offensive side b at the point M, O_c offensive side c in point N and O_d that offensive side of d at the point P. Suppose the length of radii of the circle is R_a, R_b, R_c and R_d each of which is the radii of the circle centered at O_a, O_b, O_c and O_d.

The second form of excircle is the circles are in front of the point C. Let the circle of offending extension of sides AB, BC, CD and AD, respectively at point F, I, G and H. Thus, the radii of the circle are $\rho = OH = OI = OG = OF$. Note ΔAOF, using trigonometric rules we obtained

$$AF = \rho \cot \frac{A}{2}, \tag{3.14}$$

$$\angle OBF = \frac{180^\circ - B}{2}$$

$$= 90^\circ - \frac{B}{2}.$$

So that,

$$\tan \angle OBF = \frac{\rho}{BF'},$$

$$BF = \rho \tan \frac{B}{2}. \tag{3.15}$$

By subtracting the equations (3.14) and (3.15) was obtained

$$a = \rho\left(\cot\frac{A}{2} - \tan\frac{B}{2}\right).$$

In a similar way would be obtained

$$AL = R_a \tan\frac{A}{2}, \tag{3.16}$$

$$BL = R_a \tan\frac{B}{2}. \tag{3.17}$$

By adding equations (3.16) and (3.17) was obtained

$$AL + BL = R_a\left(\tan\frac{A}{2} + \tan\frac{B}{2}\right),$$

$$a = R_a\left(\tan\frac{A}{2} + \tan\frac{B}{2}\right). \tag{3.18}$$

By doing the same way to ΔBOI and ΔCOI and ΔBO_bM and ΔCO_bM obtained

$$b = R_b\left(\tan\frac{B}{2} + \tan\frac{C}{2}\right). \tag{3.19}$$

By doing the same way to ΔDE_cG and ΔCE_cG as well as ΔCO_cN and ΔDO_cN derived

$$c = R_c\left(\tan\frac{C}{2} + \tan\frac{D}{2}\right). \tag{3.20}$$

By doing the same way to ΔAOH and ΔDOH and ΔDO_dP and ΔAO_dP obtained

$$d = R_d\left(\tan\frac{A}{2} + \tan\frac{D}{2}\right) \tag{3.21}$$

and

$$\rho\rho\left(\cot\frac{A}{2}-\tan\frac{B}{2}\right)\left(\tan\frac{B}{2}-\cot\frac{C}{2}\right)$$

$$=R_aR_b\left(\tan\frac{A}{2}+\tan\frac{B}{2}\right)\left(\tan\frac{B}{2}+\tan\frac{C}{2}\right)$$

$$=R_aR_b\left(\frac{\sin\frac{A}{2}\cos\frac{B}{2}+\cos\frac{A}{2}\sin\frac{B}{2}}{\cos\frac{A}{2}\cos\frac{B}{2}}\right)\left(\frac{\sin\frac{B}{2}\cos\frac{C}{2}+\cos\frac{B}{2}\sin\frac{C}{2}}{\cos\frac{B}{2}\cos\frac{C}{2}}\right)$$

which will be equal to

$$\rho\rho\left[\frac{\cos\frac{A+B}{2}\left(-\cos\frac{B+C}{2}\right)}{\sin\frac{A}{2}\cos^2\frac{B}{2}\sin\frac{C}{2}}\right]=R_aR_b\left[\frac{\sin\frac{A+B}{2}\sin\frac{B+C}{2}}{\cos\frac{A}{2}\cos^2\frac{B}{2}\cos\frac{C}{2}}\right],$$

$$\frac{\rho\rho}{R_aR_b}=\frac{\sin\frac{A+B}{2}\sin\frac{B+C}{2}\sin\frac{A}{2}\cos^2\frac{B}{2}\sin\frac{C}{2}}{\cos\frac{A}{2}\cos^2\frac{B}{2}\cos\frac{C}{2}\cos\frac{A+B}{2}\left(-\cos\frac{B+C}{2}\right)},$$

$$\frac{\rho\rho}{R_aR_b}=-\tan\frac{A+B}{2}\tan\frac{B+C}{2}\tan\frac{A}{2}\tan\frac{C}{2}. \tag{3.22}$$

Return the same way will be obtained

$$\frac{\rho\rho}{R_cR_d}=-\tan\frac{A+D}{2}\tan\frac{D+C}{2}\tan\frac{A}{2}\tan\frac{C}{2} \tag{3.23}$$

since

$$\tan\frac{A+B}{2}=-\tan\frac{D+C}{2},$$

and

$$\tan\frac{B+C}{2}=-\tan\frac{A+D}{2}.$$

So that

$$\tan\frac{A+B}{2}\tan\frac{B+C}{2}=\tan\frac{A+D}{2}\tan\frac{C+D}{2},$$

by equations (3.22) and (3.23) was obtained

$$\frac{\rho\rho}{R_a R_b} = \frac{\rho\rho}{R_c R_d}$$

so

$$R_a R_b = R_c R_d.$$

($\Leftarrow$) Suppose $R_a R_b = R_c R_d$ will be shown that $\square ABCD$ has excircle. Note Figure 3.4. Suppose the point of tangency of the circle centered at the point O_a, O_b, O_c and O_d that points L, M, N and P.

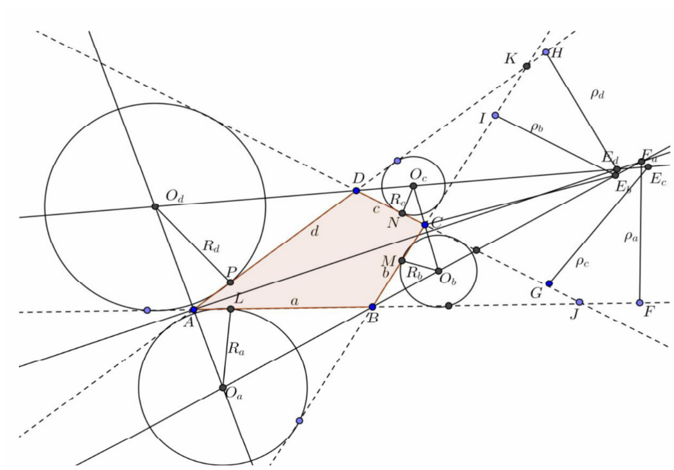

Figure 3.4

E_a point is the center point formed by the intersection of each bisector $\angle A$ and $\angle CBJ$. E_b point is the center point formed by the intersection of each bisector $\angle C$ and $\angle CBJ$. E_c point is the center point formed by the intersection of each bisector $\angle C$ and $\angle CDK$. And E_d point is the center point formed by the intersection of each bisector $\angle A$ and $\angle CDK$. ρ_a, ρ_b, ρ_c and ρ_d are the lengths of the radii of the circle, each centered on E_a, E_b, E_c and E_d and offend the extension side of a, b, c and d. So will apply

$$\frac{\rho_a \rho_b}{R_a R_b} = -\tan\frac{A+B}{2}\tan\frac{B+C}{2}\tan\frac{A}{2}\tan\frac{C}{2}.$$

As well as

$$\frac{\rho_c\rho_d}{R_cR_d} = -\tan\frac{A+D}{2}\tan\frac{D+C}{2}\tan\frac{A}{2}\tan\frac{C}{2}.$$

And since it also applies

$$\tan\frac{A+B}{2}\tan\frac{B+C}{2} = \tan\frac{A+D}{2}\tan\frac{C+D}{2},$$

then we obtain

$$\frac{\rho_a\rho_b}{R_aR_b} = \frac{\rho_c\rho_d}{R_cR_d}.$$

Since $R_aR_b = R_cR_d$, it should $\rho_a\rho_b = \rho_c\rho_d$. From the definition of ρ_a, ρ_b, ρ_c and ρ_d, then must $\rho_a = \rho_b = \rho_c = \rho_d$. Since the lengths of radii are same, it must be the center point of the circle that is $E_a = E_b = E_c = E_d$ that are in front of point C, in other words $\square ABCD$ has excircle. ■

References

[1] A Gultierrez, Go Geometri, hal 1
http://www.gogeometry.com/problem/p569_quadrilateral_excircles_tangency_point_congruence.htm. acsessed 4 August 2015. 08.35 pm

[2] A. Hess, On a circle containing the incenters of tangential quadrilaterals, Forum Geom. 14 (2014), 389-396.

[3] F. Samandace and I. Fatrascu, The Geometry of Homological Triangles, The Educatios Publisher, Inc., Ohio, 2012.

[4] H. Mowaffaq, A condition for a circumscriptible quadrilateral to be cyclic, Forum Geom. 8 (2008), 103-106.

[5] L. Hoehn, A new formula concerning the diagonals and sides of a quadrilateral, Forum Geom. 11 (2011), 211-212.

[6] Mihal Miculita, A new proferty of circumscribed quadrilatera, Internat. J. Geom. 1(2) (2012), 61-64.

[7] M. Josefsson, On the inradius of a tangential quadrilateral, Forum Geom. 10 (2010), 27-34.

[8] M. Josefsson, More characterizations of tangential quadrilateral, Forum Geom. 11 (2011), 65-82.

[9] M. Josefsson, The area of a bicentric quadrilateral, Forum Geom. 11 (2011), 155-164.

[10] M. Josefsson, Similar metric characterizations of tangential and extangential quadrilaterals, Forum Geom. 12 (2012), 63-77.

[11] N. Minculete, Characterizations of a tangential quadrilateral, Forum Geom. 9 (2009), 113-118.

[12] M. Radic, Z. Kariman and V. Kadum, A condition that a tangential quadrilateral is also a chordal one, Math. Commun. 12 (2007), 33-52.

[13] Paul Yiu, Introduction to the Geometry of the Triangle, Florida Atlantic University, 2012.