# Some Thought on Numerical Integration Based on Interpolation *

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**Abstract**

We discuss and do some analysis on numerical integration based on interpolation, midpoint, trapezoidal rule and Simpson rule. We end up with some new formulas, which are not mentioned in numerical analysis textbooks. The strategy we discuss, in terms of pedagogy, illuminate how research on mathematics can be carried out.

**Keywords:** *numerical integration, midpoint rule, trapezoidal rule, Simpson rule*

## 1 Introduction

Numerical integration is a necessary tool that student should learn aside to analytic integration tools. Common numerical tools taught in numerical analysis class to approximate

$$I := \int_a^b f(x)dx \tag{1}$$

is numerical integration based on interpolation, such as trapezoidal rule, midpoint rule and Simpson rule. If $f$ is smooth enough, trapezoidal rule formula for an approximation of a definite integral (1) is given by

$$T := \frac{b-a}{2}(f(a) + f(b)), \tag{2}$$

with its approximation error $E_T = -\frac{(b-a)^3}{12} f''(\xi)$ where $\xi \in (a,b)$ and $f \in C^2[a,b]$. This method is exact if $f$ is linear and needs two function evaluations for single use. Simpson rule to approximate (1) is given by

$$S := \frac{b-a}{6}(f(a) + 4f(\frac{a+b}{2}) + f(b)), \tag{3}$$

with its approximation error $E_S = -\frac{(b-a)^5}{90} f^{(4)}(\xi)$ where $\xi \in (a,b)$ and $f \in C^4[a,b]$. This method has an order $\mathcal{O}(h^4)$ and needs three function evaluations for single use. The midpoint rule is given by

$$M := (b-a)f(\frac{a+b}{2}), \tag{4}$$

with its approximation error $E_M = \frac{(b-a)^3}{24} f''(\xi)$ where $\xi \in (a,b)$ and $f \in C^2[a,b]$. This method is exact for a linear function and needs only one function evaluation for single use.

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Corrected trapezoidal rule to approximate (1) for $f'$ exists on $[a,b]$ is given

$$T_C := \frac{(b-a)}{2}(f(a)+f(b)) - \frac{(b-a)^2}{12}(f'(b)-f'(a)), \tag{5}$$

with $E_{TC} = \frac{(b-a)^5}{720} f^{(4)}(\xi)$ where $\xi \in (a,b)$ and $f \in C^4[a,b]$. This method has an order $\mathcal{O}(h^4)$ and needs two functions and two end derivative evaluations for single use. Corrected midpoint rule to approximate (1) for $f'$ exists on $[a,b]$ is given

$$M_C := (b-a)f(\frac{a+b}{2}) + \frac{(b-a)^2}{24}(f'(b)-f'(a)). \tag{6}$$

This method has an order $\mathcal{O}(h^4)$ and needs two functions and two end derivative evaluations for single use.[1, 2, 7]

We see that the approximation error of the midpoint rule needs the same smoothness as the trapezoidal rule. We only require the second derivative of $f$ instead of the fourth derivative of $f$ for Simpson rule. The absolute value of midpoint approximation error is slightly smaller than the trapezoidal rule, by comparing the constant in front of the derivative of $f$. However the trapezoidal rule is better for a periodic function, Engeln-Müllges & Uhlig [3]. The absolute value of corrected trapezoidal rule error is seven times smaller that the absolute value of Simpson rule error.

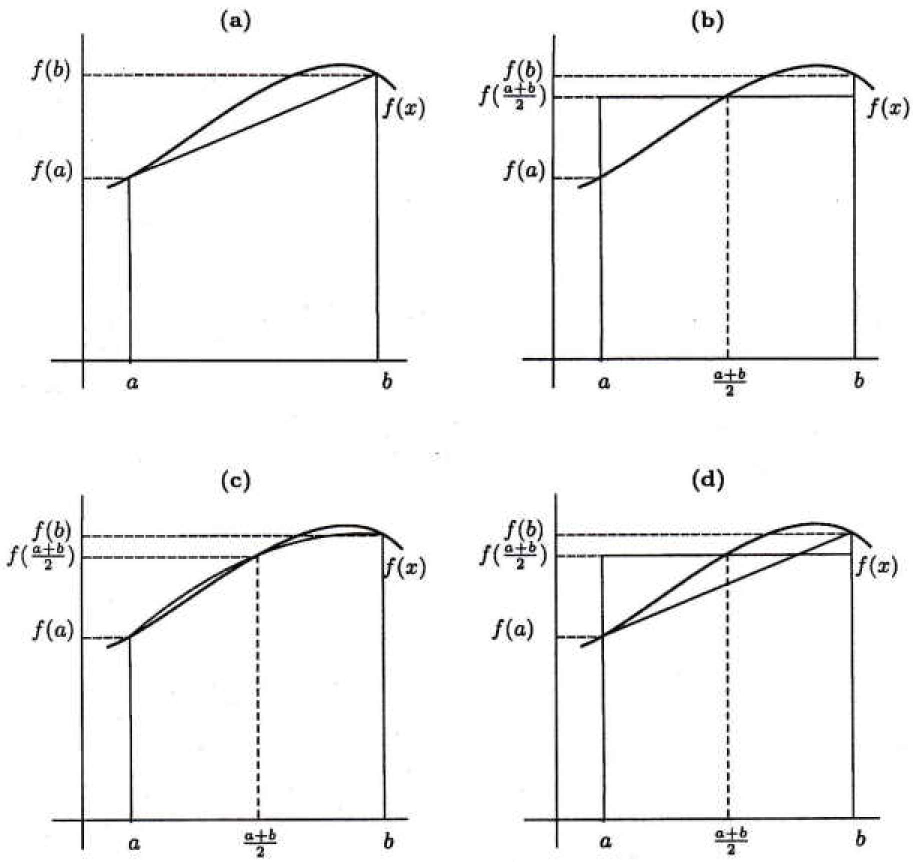


Figure 1: Graph of approximation of $\int_a^b f(x)dx$ using (a) Trapezoidal rule, (b) Midpoint rule, (c) Simpson rule, and (d) Trapezoidal and Midpoint rule

In this study we suggest a convex combination of the trapezoidal and midpoint rule to obtain a better rule to approximate (1). We derive some methods which are commonly not mentioned in numerical analysis textbooks.

## 2 Proposed Method

Now we consider a convex combination of $T$ and $M$, called it as $TM$, that is

$$TM = (1-\alpha)T + \alpha M, \quad \text{where} \quad \alpha \in [0,1]. \tag{7}$$

We use (7) to obtain a new rule that is better than $T$ and $M$. If we choose $\alpha = \frac{1}{2}$ in (7), we obtain an arithmetic mean of $T$ and $M$, call it $TM_m$, that is

$$\begin{aligned} TM_m &= \frac{1}{2}(T+M) \\ &= \frac{1}{2}\Big((b-a)\frac{(f(a)+f(b))}{2} + (b-a)f(\frac{a+b}{2})\Big) \\ &= \frac{(b-a)}{2}\Big(\frac{(f(a)+f(b))}{2} + f(\frac{a+b}{2})\Big) \\ TM_m &= \frac{(b-a)}{2}\Big(\frac{f(a)+f(\frac{a+b}{2})}{2} + \frac{f(\frac{a+b}{2})+f(b)}{2}\Big). \end{aligned} \tag{8}$$

We see that using an arithmetic mean of $T$ and $M$ is the same as applying a Trapezoidal rule twice, a composite trapezoidal rule. This means that we do not obtain a better rule to approximate a definite integral $I$.

Horwitz [4] shows that Simpson rule can be expressed in terms of this combination by choosing $\alpha = \frac{2}{3}$ in (7), that is

$$\begin{aligned} TM_{CC} &= \frac{1}{3}T + \frac{2}{3}M \\ &= \frac{1}{3}\Big((b-a)\frac{(f(a)+f(b))}{2}\Big) + \frac{2}{3}\Big((b-a)f(\frac{a+b}{2})\Big) \\ TM_{CC} &= \frac{(b-a)}{6}\Big(f(a) + 4f(\frac{a+b}{2}) + f(b)\Big) =: S. \end{aligned} \tag{9}$$

Can we obtain a better method than Simpson rule using (7), i.e. can we find $\alpha$ such that the error of $TM$ to approximate (1) is as small as possible?. To do this we do a simulation using erf function as test function, on $[0,1]$, that is

$$\int_0^1 \frac{2.0}{\sqrt{(\pi)}} \exp(-x^2)dx = 0.842700792949714869341220635082609259296 1. \tag{10}$$

Then we vary $\alpha \in [0,1]$. We do this by dividing $[0,1]$ into 150 node points and applying a composite (7) for $n = 100$. We plot the results as depicted in Figure 2. The best $\alpha$ from this simulation is $0.667 \approx \frac{2}{3}$, as can be seen in Figure 2. This means that the best method we can obtain using a convex combination of $T$ and $M$ in (7) is Simpson rule as in (9).

Now consider applying a composite trapezoidal rule to approximate (1) by dividing $[a,b]$ into $2n$ subinterval, $[x_i, x_{i+1}]$, $i = 0,1,\ldots,2n$. Here $x_0 = a$ and $x_{2n} = b$. To simplify we write $f_i = f(x_i)$, $i = 0,1,\ldots,2n$. Thus

$$T_{2n} = \frac{b-a}{4n}(f_0 + 2f_1 + 2f_2 + \cdots + 2f_{2n-2} + 2f_{2n-1} + f_{2n}). \tag{11}$$

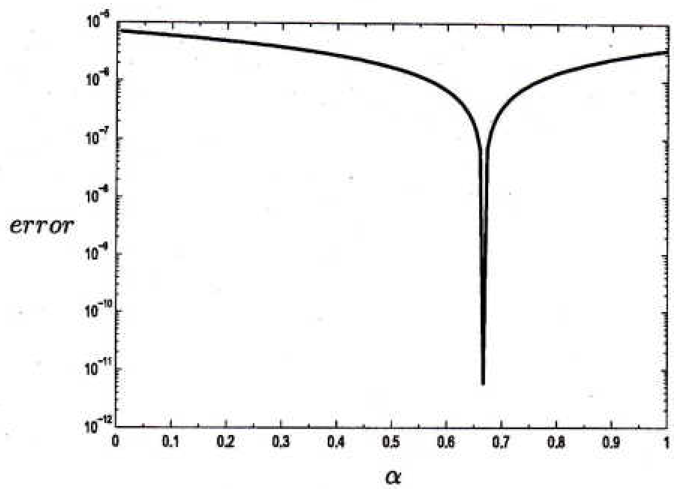


Figure 2: The best $\alpha \in [0,1]$ that has the smallest *error* for finding a best convex combination of $T$ and $M$

In the same way, we can obtain a composite trapezoidal rule in $n$ subintervals, that is

$$T_n = \frac{b-a}{2n}(\bar{f}_0 + 2\bar{f}_1 + 2\bar{f}_2 + \cdots + 2\bar{f}_{n-2} + 2\bar{f}_{n-1} + \bar{f}_n). \tag{12}$$

Here $\bar{f}_i = f_{2i}$, $i = 0, 1, \ldots, n$. If we multiply (11) by $\frac{4}{3}$ and (12) by $\frac{1}{3}$ respectively, we obtain

$$\frac{4}{3}T_{2n} = \frac{b-a}{6n}(2f_0 + 4f_1 + 4f_2 + \cdots + 4f_{2n-2} + 4f_{2n-1} + 2f_{2n}) \tag{13}$$

and

$$\frac{1}{3}T_n = \frac{1}{2}\frac{b-a}{6n}(f_0 + 2f_2 + 2f_4 + \cdots + 2f_{2n-4} + 2f_{2n-2} + f_{2n}). \tag{14}$$

Subtracting (14) from (13) and call it as $S_n$, we obtain

$$\begin{aligned} S_n &= \frac{4}{3}T_{2n} - \frac{1}{3}T_n \\ &= \frac{b-a}{6n}(f_0 + 4f_1 + 2f_2 + \cdots + 2f_{2n-2} + 4f_{2n-1} + f_{2n}) \\ S_n &= \frac{h}{3}(f_0 + 4f_1 + 2f_2 + \cdots + 2f_{2n-2} + 4f_{2n-1} + f_{2n}) \end{aligned} \tag{15}$$

where $h = \frac{b-a}{2n}$ and the value of $f_i$ are multiplied by 4 and 2 alternately, for $i = 1, 2, \ldots, 2n-1$. Thus we obtain a composite Simpson rule [7]. Following this idea and noting (9), we can also obtain a composite Simpson rule, as combination of $T_n$ and $M_n$, a composite midpoint rule, as follows

$$S_n = \frac{1}{3}(2T_{2n} + M_n).$$

Following the idea obtaining Simpson rule, can we obtain a corrected simpson rule, from a convex combination of the corrected trapezoidal rule, $T_C$, and the corrected midpoint rule, $M_C$?. To answer this question, we write

$$TM_C = (1-\alpha)T_C + \alpha M_C. \tag{16}$$

If we take $\alpha = \frac{2}{3}$ as Horwitz [4] use we obtain $(1-\alpha)T_C$, as follows

$$\frac{1}{3}T_C = \frac{(b-a)}{6}(f(a)+f(b)) - \frac{(b-a)^2}{36}(f'(b)-f'(a)), \tag{17}$$

and

$$\frac{2}{3}M_C = \frac{2}{3}(b-a)f(\frac{a+b}{2}) + \frac{(b-a)^2}{36}(f'(b)-f'(a)). \tag{18}$$

Adding (17) into (18) and simplifying, we end up with

$$\begin{aligned} TM_{C1} &= \frac{1}{3}T_C + \frac{2}{3}M_C \\ &= \frac{(b-a)}{6}(f(a)+f(b)) + \frac{2}{3}(b-a)f(\frac{a+b}{2}) \\ TM_{C1} &= \frac{(b-a)}{6}(f(a)+4f(\frac{a+b}{2})+f(b)) =: S. \end{aligned} \tag{19}$$

Thus we obtain again $S$, which is not better than $T_C$ or $M_C$.

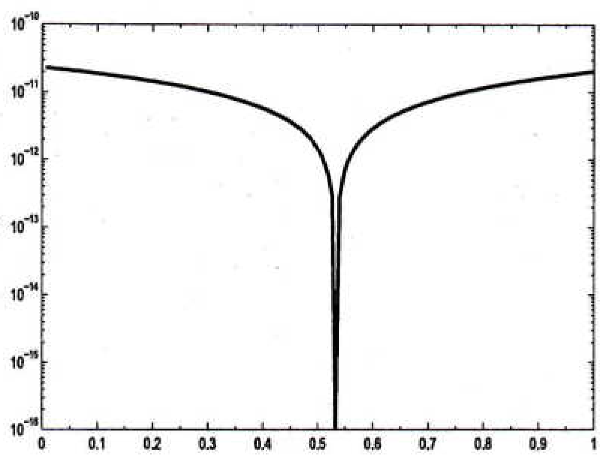


Figure 3: The best $\alpha \in [0,1]$ that has the smallest *error* for finding a best convex combination of $T_C$ and $M_C$

Now we play the same strategy as finding a best convex combination ending up with Simpson rule. We use erf function, (10), as test function. Then, we vary $\alpha \in [0,1]$. We do this by dividing $[0,1]$ into 150 node points and applying a composite (16) for $n = 100$. We plot the results as depicted in Figure 3. We see that $\alpha = 0.533 \approx \frac{8}{15}$ which has a smallest error. Inserting this $\alpha$ into (16), we have

$$\begin{aligned} TM_{CC} &= \frac{7}{15}T_C + \frac{8}{15}M_C \\ &= \frac{7}{15}\Big(\frac{(b-a)}{3}(f(a)+f(b)) - \frac{(b-a)^2}{12}(f'(b)-f'(a))\Big) \\ &\quad + \frac{8}{15}\Big((b-a)f(\frac{a+b}{2}) + \frac{(b-a)^2}{24}(f'(b)-f'(a))\Big) \\ &= \frac{(b-a)}{3}(7f(a)+16f(\frac{a+b}{2})+7f(b)) - \frac{(b-a)^2}{60}(f'(b)-f'(a)) =: S_C. \end{aligned} \tag{20}$$

We find a corrected Simpson like rule, as obtained by Liu [6].

We remark that it is also possible to derive Simpson rule via geometry, as discussed by Kendig [5] and Richardson [8].

## 3 Numerical Experiments

In this section we do some computation to compare the accuracy of the method we discuss in the previous section. We use erf function, (10), as test function. We use the same numbers of interval to compare the accuracy of the methods, that is a single Simpson rule we compare with twice trapezoidal rule, and midpoint rule, etc. We use the number of interval $2n$, $n = 30, 60$. The comparison results is as depicted in Table 1.

Table 1: Numerical results for approximation of erf function using some discuss methods

| Methods | Intervals | $n - composite$ | Approximations | *Error* |
|---|---|---|---|---|
| $T$ | 60 | 60 | 0.8426815748320778 | 0.0000192181176371 |
| $M$ | 60 | 60 | 0.8427104020752612 | 0.0000096091255463 |
| $S$ | 60 | 30 | 0.8427007936614431 | 0.0000000007117282 |
| $T_C$ | 60 | 60 | 0.8427007927717737 | 0.0000000001779412 |
| $M_C$ | 60 | 60 | 0.8427007931054136 | 0.0000000001556987 |
| $S_C$ | 60 | 30 | 0.8427007929497075 | 0.0000000000000074 |
| $T$ | 120 | 120 | 0.8426959884536690 | 0.0000048044960459 |
| $M$ | 120 | 120 | 0.8427031952019083 | 0.0000024022521934 |
| $S$ | 120 | 60 | 0.8427007929941998 | 0.0000000000444849 |
| $T_C$ | 120 | 120 | 0.8427007929385929 | 0.0000000000111220 |
| $M_C$ | 120 | 120 | 0.8427007929594463 | 0.0000000000097314 |
| $S_C$ | 120 | 60 | 0.8427007929497147 | 0.0000000000000002 |

From Table 1 we see that the results match the preliminaries results stated in the introduction. The simulations also show that the corrected Simpson like method is better than the other discussed methods. However, all corrected methods can not be applied for the function having no derivatives at the end of intervals, such as $f(x) = \sqrt{x}$ on $[a, b] = [0, 1]$.

## 4 Pedagogical Notes

The ideas discuss in this article could be used as a project for numerical analysis classes. This also forms the basis for a student to understand how to do research in mathematics, especially deriving a new numerical method. Doing such thing can encourage student to learn a higher mathematics level based on the material they are familiar to.

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