A ROTARY HEURISTIC FOR LOCATION-ALLOCATION PROBLEMS

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Abstract

This paper proposes a constructive heuristic method to solve location-allocation problems. Specifically, we consider the problem of locating m new facilities in a continuous region such that the sum of the weighted distances from the new facilities to n existing facilities is minimized. The distance is measured using the Euclidean-distance metric. This simple technique shows that the solution found is encouraging for the case where the number of users is much larger than the number of facilities to be located.

Keywords: facility location, location-allocation, heuristic.

1. Introduction

In this study, we are given a set of users, located at n fixed points, with their respective demands. We are required to locate m new facilities in continuous space to serve these n users, and to find the allocation of these users to these m facilities. The objective is to minimize the sum of transportation costs. This continuous location-allocation problem is also known as the multi-source Weber problem.

In this problem, it is assumed that (i) the new facilities are independent from each other; (ii) the number of new facilities is given; (iii) the cost function is proportional to the product of the quantity and the Euclidean distance between new facilities and users; (iv) the facilities to open have infinite capacity; *i.e.* the capacity constraints are ignored.

The location-allocation problems in the continuous space $\Re^2$ can be formulated as follows:

$$\min_{W,X} \sum_{i=1}^{m}\sum_{j=1}^{n} w_{ij} \left\|\mathbf{X}_i - \mathbf{A}_j\right\| \tag{1}$$

$$\text{subject to} \quad \sum_{i=1}^{m} w_{ij} = r_j, \quad j = 1, 2, \ldots, n, \tag{2}$$

$$w_{ij} \geq 0, \quad i = 1, 2, \ldots, m; \quad j = 1, 2, \ldots, n, \tag{3}$$

where input data consist of

- $\mathbf{A}_j = (x_j, y_j)$, location of user j, $j = 1, 2, \ldots, n$,

- $r_j \geq 0$, demand rate of user j, $j = 1, 2, \dots, n$,

and the decision variables are

- $\mathbf{X} = (\mathbf{X}_1, \dots, \mathbf{X}_m)$, vector of location variables, where $\mathbf{X}_i = (X_i, Y_i)$ is location of facility i to be determined; $i = 1, 2, \dots, m$.
- $\mathbf{W} = (w_{ij})$, vector of allocation variables, where w_{ij} denotes the flow from user j served by facility i; $i = 1, 2, \dots, m$, $j = 1, 2, \dots, n$,
- and $\|\mathbf{X}_i - \mathbf{A}_j\| = \sqrt{(X_i - x_j)^2 + (Y_i - y_j)^2}$ is the Euclidean distance between facility i and user j.

Constraint (2) guarantees that the demand at each user can be satisfied. Constraints (3) refer to the non-negativity of the decision variables. Under the assumption that there are no capacity constraints at the new facilities, it can be shown that the demand at each user ia satisfied in the minimum cost by the nearest facility. For the number of facilities more than one ($m > 1$) the objective function (1) is neither concave nor convex, and may contain several local minima (Cooper, 1964). Hence, the continuous location-allocation problem falls in the realm of NP-hard optimisation problems.

1. Previous Works

Many heuristic methods have been proposed in the literature beginning with the well-known iterative location-allocation algorithm of Cooper (1964) to solve the multi-source Weber problem. Cooper's heuristic generates p subsets of user points and then solves each one optimally using Weiszfeld iterative method for solving a single-facility location problem. In the following section, we will discuss Weiszfeld iterative method since this method will be used as a basis in the design of the heuristics which we are putting forward in this paper. Kuenne and Soland (1972) create a branch-and-bound algorithm which produces an exact solution for problems with 25 user points and 1 to 5 facilities. Love and Morris (1975) develop the set reduction method and a p-median algorithm to solve the multi-source Weber problem with rectangular distance. Their method gives the exact solution to problems with 35 user points and 2 facilities. Rosing (1992) proposes a method to solve the (generalized) multi-source Weber problem. He divides the set of user points into non-overlapping convex hulls and generates the list of all feasible convex hulls where each user point must belong to exactly one of those convex hulls. The cost function associated with each convex hull is computed as a single Weber problem. This method produces the optimal solution to problems with up to 30 user points and 6 facilities.

There are several heuristic methods introduced to solve problems of larger size. Gamal and Salhi (2001) presented a constructive heuristic based on the furthest distance rule to find initial locations while introducing forbidden regions to avoid locations being too close to each others. Gamal and Salhi (2003) created a discretisation based approach known as a cellular heuristic. Brimberg *et. al.* (2013) propose an effective constructive heuristic that find a good initial solution by combining the drop method and the gravity concept. Very recently, Brimberg *et. al.* (2014) present a new local search for solving continuous location problems based on reformulations of the problem in continuous and discrete space. They first conduct a local search in the continuous space until a local optimum is obtained. Then they augment a specified set of user points with the local optima obtained in the continuous space. With this augmenting nodes of network, user points plus local optima, they solve a discrete problem.

Until now heuristic method is still a popular method to solve large scale location-allocation problems and to solve many types of combinatoric optimization problems. In this paper we propose a simple heuristic technique that divides the user set consructively using line divisors centered at the center of gravity point as a rotation center point. Other simple techniques can also be utilized to get the center point. Centering at the rotation point, these line divisors are rotated by angle α arbitrarily to create some user subsets.

3. Weiszfeld's Iterative Method

First we divide the user set into m user subsets. For each of these subsets, we use an exact method to find an optimal location for single facility. The iterative method of Weiszfeld (1937) is an exact method to solve single-facility location problems. Let superscript k denote the number of iterations. Then the iterative method is given by equation (3).

$$X^{(k)} = \frac{\sum_{j=1}^{n} \frac{r_j x_j}{\left\| \mathbf{X}^{(k-1)} - \mathbf{A}_j \right\|}}{\sum_{j=1}^{n} \frac{r_j}{\left\| \mathbf{X}^{(k-1)} - \mathbf{A}_j \right\|}} , \qquad Y^{(k)} = \frac{\sum_{j=1}^{n} \frac{r_j y_j}{\left\| \mathbf{X}^{(k-1)} - \mathbf{A}_j \right\|}}{\sum_{j=1}^{n} \frac{r_j}{\left\| \mathbf{X}^{(k-1)} - \mathbf{A}_j \right\|}} . \qquad (3)$$

Thus the initial location $\left(X^{(0)}, Y^{(0)}\right)$ is needed to obtain $\left(X^{(1)}, Y^{(1)}\right)$. The value $\left(X^{(1)}, Y^{(1)}\right)$ is then used to obtain $\left(X^{(2)}, Y^{(2)}\right)$, and so on. This process is convergent in a subset and the minimum value obtained is a local minimum.

If we want to have an exact solution by way of dividing the user set into some user subsets, all we have to do is to carry out a complete enumeration; *i.e.* by generating all possible combinations. Suppose that there are n user fixed points and m facilities to be located to serve these n users. The number of all possible user subsets is equivalent to the number of all possible partitions of a set of size n partitioned into m nonempty subsets. This number follows Stirling's number of the second type, that is

$$S(m,n)=\frac{1}{n!}\sum_{k=0}^{n}\binom{n}{k}(-1)^k(n-k)^m$$

For example, to have a complete enumeration for m = 2 and n = 50 we have S(2,50) = 562,949,953,421,311 possible user subsets. However, some of the subsets make no sense from the optimization view of points. Due to this large number, the heuristic methods are still the sophisticated approaches to generate the user subsets.

4. Rotary Heuristic

This heuristic technique divides the user region constructively using a given number of line divisors centered at a rotation point. Centering at the rotation point, these lines are rotated by any angle α. For more details, suppose we have the case of locating two facilities to serve 50 users with their location coordinates presented in Table 1.

Table 1. The 50-user problem (Eilon *et al.*, 1971)

User No.	User Location (x, y)		User No.	User Location (x, y)	
	x	y		x	y
1	1.33	8.89	26	4.46	7.91
2	1.89	0.77	27	2.83	9.88
3	9.27	1.49	28	3.39	5.65
4	9.46	9.36	29	0.75	4.98
5	9.20	8.69	30	7.55	5.79
6	7.43	1.61	31	8.45	0.69
7	6.08	1.34	32	3.33	5.78
8	5.57	4.60	33	6.27	3.66
9	6.70	2.77	34	7.31	1.61
10	8.99	2.45	35	6.37	7.02
11	8.93	7.00	36	7.23	7.05
12	8.60	0.53	37	1.68	6.45
13	4.01	0.31	38	3.54	7.06
14	3.34	4.01	39	7.67	4.17
15	6.75	5.57	40	2.20	1.12
16	7.36	4.03	41	3.57	1.99
17	1.24	6.69	42	7.34	1.38
18	3.13	1.92	43	6.58	4.49
19	8.86	8.74	44	5.00	9.00
20	4.18	3.74	45	6.63	5.23
21	2.22	4.53	46	5.89	8.06
22	0.88	7.02	47	1.13	5.25
23	8.53	7.04	48	1.90	8.35
24	6.49	6.22	49	1.74	1.37
25	4.53	7.87	50	9.39	6.44

The user set combinations are selected by dividing the rectangular region that covers all the user points into two parts using a straight line rotated many times. The straight line is rotated at a rotation point as illustrated in Figure 1, that is the point obtained by intersection of two diagonals being formed by four user corner points in the perpendicular region. The four corner points are $T_1\,(0.75, 0.31)$, $T_2\,(9.46, 0.31)$, $T_3\,(9.46, 9.88)$, and $T_4\,(0.75, 9.88)$, while the rotation center is $T\,(5.11, 5.10)$. The straight lines divide the region into two sub-regions or two user subsets

Figure 1 shows that Line 1 divides the users into two user subsets, that is {1, 2, 13, 14, 17, 18, 20, 21, 22, 24, 26, 27, 28, 29, 32, 37, 38, 40, 41, 44, 47, 48, 49} and {3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 15, 16, 19, 23, 24, 30, 31, 33, 34, 35, 36, 39, 42, 43, 45, 46, 50}; Line 2 divides the users into subsets {1, 2, 14, 17, 18, 20, 21, 22, 25, 26, 27, 28, 29, 32, 37, 38, 40, 41, 44, 46, 47, 48, 49} and { 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 15, 16, 19, 23, 24, 30, 31, 33, 34, 35, 36, 39, 42, 43, 45, 50}.

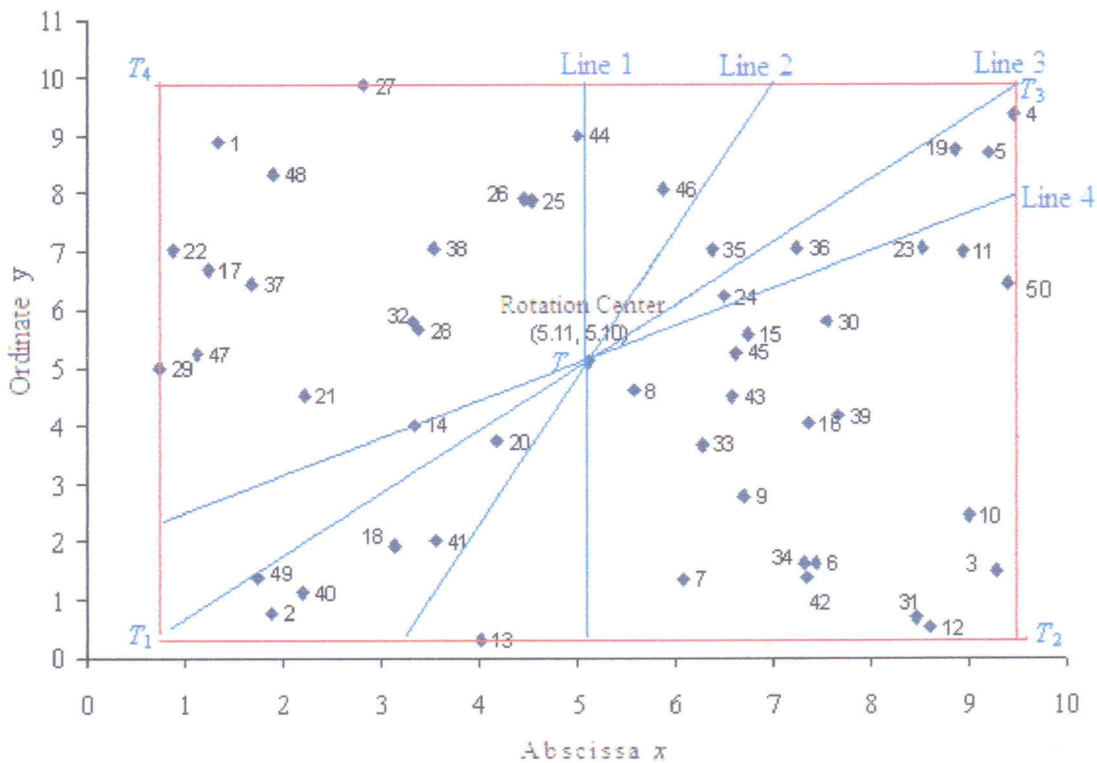


Figure 1. Location coordinates of 50 users

Each subset performed is considered as a problem of locating a single facility where the solution can be found using equation (3). The solution obtained for each subset is an exact one. For the starting solution $\left(X^{(0)}, Y^{(0)}\right)$ we can choose a point at random in each of the subsets. In order to speed up the convergency of this iterative method using equations (3) we should use the center of gravity method to obtain the initial solution (Eilon *et al.*, 1971). The center of gravity method is given by the following equations:

$$X^{(0)} = \frac{\sum_{j=1}^{n} r_j x_j}{\sum_{j=1}^{n} r_j}, \qquad Y^{(0)} = \frac{\sum_{j=1}^{n} r_j y_j}{\sum_{j=1}^{n} r_j} \tag{4}$$

In general finding the solution of location allocation problems using the rotary heuristic method takes some steps as follows:

Step 1 Create a combination of user subsets as many as the number of facilities to be located, in this experiment let $m = 2$, using a line divisor.

Step 2 For each of these m non-overlapping user subsets, find the exact optimal solution using equations (3) whose initial solution is obtained using equation (4). Here we get some locally optimal solutions. Then obtain the total transportation cost by summing the transportation cost of each user subset.

Step 3 Rotate the line divisor at rotation point by any angle α to obtain other combinations of new user subsets. Then repeat Step 2.

Step 4 Repeat Step 2 and Step 3 several times, take the combination of the user subset with the minimum total transportation cost as global solution.

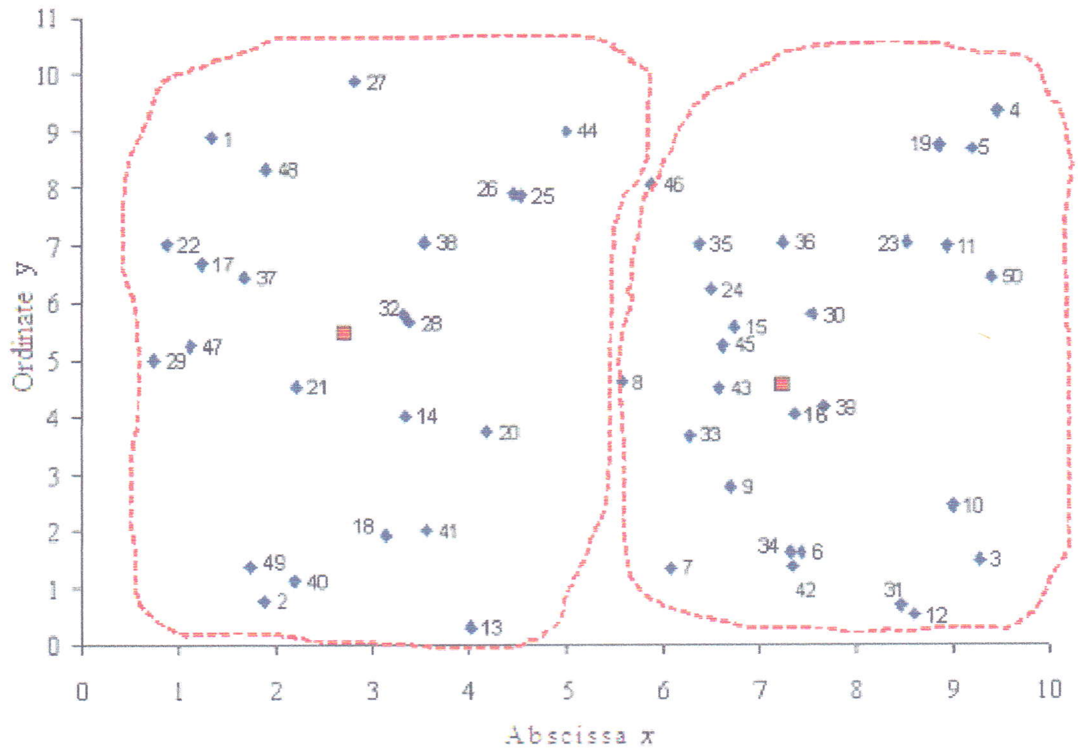
It was mentioned before that the solution globally obtained is not an optimal one. In order to get a solution near the optimal one, generating the user subsets needs to be repeated several times.

5. Hasil Eksperimen Permulaan

Pada eksperimen ini diambil $m = 2$ dan $r_j = 1$ untuk setiap pengguna j. Sudut rotasi diambil $\alpha = 5^0$ searah dengan jarum jam. Untuk komputasi digunakan MATLAB versi 5.3. Hasil komputasi dapat dilihat pada Tabel 1.

Berdasarkan Tabel 3.1 dapat dilihat bahwa dari 15 kombinasi yang diambil, diperoleh biaya minimum pada Kombinasi 1, yakni dengan total biaya transportasi 135.546. Dari hasil ini diperoleh bahwa pelanggan nomor 1, 2, 13, 14, 17, 18, 20, 21, 22, 24, 26, 27, 28, 29, 32, 37, 38, 40, 41, 44, 47, 48, dan 49 dialokasikan ke fasilitas 1 yang terletak pada koordinat (2.691 , 5.480), sedangkan pelanggan nomor 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 15, 16, 19, 23, 24, 30, 31, 33, 34, 35, 36, 39, 42, 43, 45, 46, 50 dialokasikan ke fasilitas 2 yang terletak pada koordinat (7.280 , 4.689).

Solusi yang diperoleh pada eksperimen ini menyimpang sebesar 0,019% dari solusi optimal dengan biaya transportasi sebesar 135.52 (Eilon *et al.*, 1971), tetapi sedikit lebih baik dari metode *subset destination* Cooper (Cooper, 1964) yang memperoleh biaya transportasi 135.552 dengan koordinat fasilitas pada (2.67, 5.65) dan (7.24, 4.54). Gambaran penyebaran koordinat pengguna dan lokasi fasilitas yang akan dibangun dapat dilihat pada Gambar 2.



Notes:

■ = Coordinates of local optimal facility locations

♦ = User coordinates

Figure 2. Location of two facilities to serve two user subsets

Table 1. Experimental results for 15 user subsets

No.	Subsets of Users	Facility Locations	Total Cost
1.	{1, 2, 13, 14, 17, 18, 20, 21, 22, 24, 26, 27, 28, 29, 32, 37, 38, 40, 41, 44, 47, 48, 49} ; {3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 15, 16, 19, 23, 24, 30, 31, 33, 34, 35, 36, 39, 42, 43, 45, 46, 50}	(2.701 , 5.476) (7.280 , 4.689)	135.546
2.	{1, 2, 14, 17, 18, 20, 21, 22, 25, 26, 27, 28, 29, 32, 37, 38, 40, 41, 44, 46, 47, 48, 49} ; { 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 15, 16, 19, 23, 24, 30, 31, 33, 34, 35, 36, 39, 42, 43, 45, 50}	(2.824 , 5.741) (7.300 , 4.358)	135.592

3.	{1, 2, 14, 17, 18, 20, 21, 22, 25, 26, 27, 28, 29, 32, 37, 38, 40, 44, 46, 47, 48, 49} ; {3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 15, 16, 19, 23, 24, 30, 31, 33, 34, 35, 39, 40, 41, 42, 43, 45, 50}	(2.776 , 5.889) (7.233 , 4.295)	136.064
4.	{1, 14, 17, 21, 22, 25, 26, 27, 28, 29, 32, 36, 37, 38, 44, 46, 47, 48, 49} ; {2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 15, 16, 18, 19, 20, 23, 24, 30, 31, 33, 34, 35, 39, 40, 41, 42, 43, 45, 50}	(2.871 , 6.609) (6.869 , 4.013)	139.175
5.	{1, 4, 5, 14, 17, 19, 21, 22, 24, 25, 26, 27, 28, 29, 32, 35, 36, 37, 38, 44, 46, 47, 48} ; {2, 3, 6, 7, 8, 9, 10, 11, 12, 13, 15, 16, 18, 20, 23, 30, 31, 33, 34, 39, 40, 41, 42, 43, 45, 49, 50}	(3.724 , 7.072) (6.517 , 3.254)	146.069
6	{1, 4, 5, 17, 19, 21, 22, 23, 24, 25, 26, 27, 28, 29, 32, 35, 36, 37, 38, 44, 46, 47, 48}; {2, 3, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 18, 20, 30, 31, 33, 34, 39, 40, 41, 42, 43, 45, 49, 50}	(4.171 , 7.357) (6.493 , 3.074)	140.277
7.	{1, 4, 5, 11, 15, 17, 19, 22, 23, 24, 25, 26, 27, 28, 29, 30, 32, 35, 36, 37, 38, 44, 46, 47, 48, 50}; {2, 3, 6, 7, 8, 9, 10, 12, 13, 14, 16, 18, 20, 21, 31, 33, 34, 39, 40, 41, 42, 43, 45, 49}	(4.973 , 7.344) (6.164 , 2.632)	140.897
8.	{1, 4, 5, 11, 15, 17, 19, 22, 23, 24, 25, 26, 27, 28, 30, 32, 35, 36, 37, 38, 44, 45, 46, 47, 48, 50}; {2, 3, 6, 7, 8, 9, 10, 12, 13, 14, 16, 18, 20, 21, 29, 31, 33, 34, 39, 40, 41, 42, 43,49}	(5.404 , 7.237) (5.937 , 2.547)	141.528
9.	{1, 4, 5, 11, 15, 17, 19, 22, 23, 24, 25, 26, 27, 28, 30, 32, 35, 36, 37, 38, 44, 45, 46, 48, 50}; {2, 3, 6, 7, 8, 9, 10, 12, 13, 14, 16, 18, 20, 21, 29, 31, 33, 34, 39, 40, 41, 42, 43, 47, 49}	(5.624 , 7.249) (5.732 , 2.630)	142.109
10.	{1, 4, 5, 11, 15, 16, 19, 23, 24, 25, 26, 27, 30, 35, 36, 38, 39, 43, 44, 45, 46, 48, 50}; {2, 3, 6, 7, 8, 9, 10, 12, 13, 14, 17, 18, 20, 21, 22, 28, 29, 31, 32, 33, 34, 37, 40, 41, 42, 47, 49}	(6.621 , 6.855) (4.278 , 3.160)	144.498
11.	{1, 4, 5, 10, 11, 15, 16, 19, 23, 24, 25, 26, 27, 30, 35, 36, 38, 39, 43, 44, 45, 46, 48, 50}; {2, 3, 6, 7, 8, 9, 12, 13, 14, 17, 18, 20, 21, 22, 28, 29, 31, 32, 33, 34, 37, 40, 41, 42, 47, 49}	(6.712 , 6.715) (4.122 , 3.230)	144.656
12.	{3, 4, 5, 8, 10, 11, 15, 16, 19, 23, 24, 25, 26, 27, 30, 33, 35, 36, 38, 39, 43, 44, 45, 46, 50}; {1, 2, 6, 7, 9, 12, 13, 14, 17, 18, 20, 21, 22, 28, 29, 31, 32, 34, 37, 40, 41, 42, 47, 48, 49}	(6.894 , 6.193) (3.488 , 3.650)	144.921
13.	{3, 4, 5, 6, 8, 9, 10, 11, 12, 15, 16, 19, 23, 24, 25, 26, 27, 30, 31, 33, 34, 35, 36, 39, 42, 43, 44, 45, 46, 50}; {1, 2, 7, 13, 14, 17, 18, 20, 21, 22, 28, 29, 32, 37,	(7.018 , 5.401) (2.457 , 4.594)	138.781

	38, 40, 41, 47, 48, 49}		
14.	{3, 4, 5, 6, 8, 9, 10, 11, 12, 15, 16, 19, 23, 24, 25, 26, 30, 31, 33, 34, 35, 36, 39, 42, 43, 44, 45, 46, 50}; {1, 2, 7, 13, 14, 17, 18, 20, 21, 22, 27, 28, 29, 32, 37, 38, 40, 41, 47, 48, 49}	(7.089 , 5.305) (2.480 , 4.763)	137.811
15.	{3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 15, 16, 19, 23, 24, 30, 31, 33, 34, 35, 36, 39, 42, 43, 44, 45, 46, 50}; {1, 2, 13, 14, 17, 18, 20, 21, 22, 25, 26, 27, 28, 29, 32, 37, 38, 40, 41, 47, 48, 49}	(7.291 , 4.881) (2.582 , 5.304)	135.847

6 Kesimpulan

Metode heuristik dengan cara membagi pengguna menjadi sub-subhimpunan telah dibahas. Hasil sementara yang diperoleh untuk dua fasilitas cukup memuaskan. Teknik ini masih bisa dikembangkan untuk fasilitas lebih dari dua. Tetapi apabila jumlah fasilitas relatif besar, subhimpunan pengguna berada pada daerah berbentuk juring sempit yang menyebabkan pengguna pada subhimpunan itu berada pada posisi relatif sejajar. Eksperimen lanjut perlu dilakukan untuk menyelidiki kasus ini.

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