Univ. Beograd. Publ. Elektrotehn. Fak.
Ser. Mat. 16 (2005), 48–54.
Available electronically at **http: //pefmath.etf.bg.ac.yu**

ON VOLUMES OF n-DIMENSIONAL PARALLELEPIPEDS IN ℓ^p SPACES

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Given a linearly independent set of n vectors in a normed space, we are interested in computing the "volume" of the n-dimensional parallelepiped spanned by them. In ℓ^p ($1 \le p < \infty$), we can use the known semi-inner product and obtain, in general, $n!$ ways of doing it, depending on the order of the vectors. We show, however, that all resulting "volumes" satisfy one common inequality.

1. INTRODUCTION

On a normed space $(X, \|\cdot\|)$, the functional $g : X^2 \to \mathbb{R}$ defined by the formula

$$g(x,y) := \frac{\|x\|}{2}\big(\lambda_+(x,y) + \lambda_-(x,y)\big),$$

where

$$\lambda_\pm(x,y) := \lim_{t\to\pm 0} t^{-1}\big(\|x+ty\| - \|x\|\big),$$

satisfies the following properties:

(a) $g(x,x) = \|x\|^2$ for all $x \in X$;

(b) $g(\alpha x, \beta y) = \alpha\beta g(x,y)$ for all $x, y \in X$, $\alpha, \beta \in \mathbb{R}$;

(c) $g(x, x+y) = \|x\|^2 + g(x,y)$ for all $x, y \in X$;

(d) $|g(x,y)| \le \|x\|\|y\|$ for all $x, y \in X$.

If, in addition, the functional $g(x,y)$ is linear in $y \in X$, it is called a *semi-inner product* on X (see [**3**, **4**]). For instance, the functional

$$(1)\qquad g(x,y) := \|x\|_p^{2-p} \sum_k |x_k|^{p-1} \mathrm{sgn}\,(x_k) y_k, \quad x = (x_k),\ y = (y_k) \in \ell^p,$$

2000 Mathematics Subject Classification: 46B20, 46B45, 46C50, 46C99
Keywords and Phrases: n-dimensional parallelepipeds, semi-inner products, orthogonal projection in normed spaces, n-norms, ℓ^p spaces

defines a semi-inner product on the space ℓ^p of p-summable sequences of real numbers, for $1 \leq p < \infty$. (Here $\|\cdot\|_p$ is the usual norm on ℓ^p.)

Using a semi-inner product g, one may define the notion of orthogonality on X. In particular, we can define

$$x \perp_g y \Leftrightarrow g(x, y) = 0.$$

(Note that since g is in general not commutative, $x \perp_g y$ does not imply that $y \perp_g x$.) Further, one can also define the *g-orthogonal projection of y on x* by

$$y_x := \frac{g(x, y)}{\|x\|^2}\, x,$$

and call $y - y_x$ the *g-orthogonal complement of y on x*. Notice here that $x \perp_g y - y_x$.

In general, given a vector $y \in X$ and a subspace $S = \text{span}\{x_1, \ldots, x_k\}$ of X with $\Gamma(x_1, \ldots, x_k) := \det(g(x_i, x_j)) \neq 0$, we can define the g-orthogonal projection of y on S by

$$y_S := -\frac{1}{\Gamma(x_1, \ldots, x_k)} \begin{vmatrix} 0 & x_1 & \ldots & x_k \\ g(x_1, y) & g(x_1, x_1) & \ldots & g(x_1, x_k) \\ \vdots & \vdots & \ddots & \vdots \\ g(x_k, y) & g(x_k, x_1) & \ldots & g(x_k, x_k) \end{vmatrix},$$

for which its orthogonal complement $y - y_S$ is given by

$$y - y_S = \frac{1}{\Gamma(x_1, \ldots, x_k)} \begin{vmatrix} y & x_1 & \ldots & x_k \\ g(x_1, y) & g(x_1, x_1) & \ldots & g(x_1, x_k) \\ \vdots & \vdots & \ddots & \vdots \\ g(x_k, y) & g(x_k, x_1) & \ldots & g(x_k, x_k) \end{vmatrix}.$$

Observe here that $x_i \perp_g y - y_S$ for each $i = 1, \ldots, k$.

Next, given a finite sequence of linearly independent vectors $x_1, \ldots, x_n$ ($n \geq 2$) in X, we can construct a *left g-orthogonal sequence* $x_1^*, \ldots, x_n^*$ as in [**4**]: Put $x_1^* := x_1$ and, for $i = 2, \ldots, n$, let

$$x_i^* := x_i - (x_i)_{S_{i-1}}, \tag{2}$$

where $S_{i-1} = \text{span}\{x_1^*, \ldots, x_{i-1}^*\}$. Then clearly $x_i^* \perp_g x_j^*$ for $i, j = 1, \ldots, n$ with $i < j$. Having done so, we may now define the "volume" of the n-dimensional parallelepiped spanned by $x_1, \ldots, x_n$ in X to be

$$V(x_1, \ldots, x_n) := \prod_{i=1}^{n} \|x_i^*\|. \tag{3}$$

Due to the limitation of g, however, $V(x_1, \ldots, x_n)$ may not be invariant under permutations of $(x_1, \ldots, x_n)$.

In the following section, we shall consider the parallelepipeds spanned by n linearly independent vectors in ℓ^p $(1 \le p < \infty)$. Our main result shows that their "volumes" satisfy one common inequality, which involves the natural n-norm of those vectors in ℓ^p.

2. MAIN RESULT

Suppose, hereafter, that $1 \le p < \infty$. The so-called (natural) *n-norm* on ℓ^p is the functional $\|\cdot, \dots, \cdot\|_p : (\ell^p)^n \to \mathbb{R}$ defined by the formula

$$\|x_1, \dots, x_n\|_p := \left(\frac{1}{n!} \sum_{j_n} \cdots \sum_{j_1} \left| \begin{vmatrix} x_{1j_1} & \dots & x_{1j_n} \\ \vdots & \ddots & \vdots \\ x_{nj_1} & \dots & x_{nj_n} \end{vmatrix} \right|^p \right)^{1/p}$$

(see [1]). (Here the outer $|\cdots|$ denotes the absolute value, while the inner $|\cdots|$ denotes the determinant.) For $p = 2$, we have $\|x_1, \dots, x_n\|_2 = \sqrt{\det(\langle x_i, x_j \rangle)}$, which represents the Euclidean volume of the n-dimensional parallelepiped spanned by $x_1, \dots, x_n$ in ℓ^2. (Here $\langle \cdot, \cdot \rangle$ denotes the usual inner product on ℓ^2.) For $n = 1$, the 1-norm coincides with the usual norm on ℓ^p. The n-norm $\|\cdot, \dots, \cdot\|_p$ on ℓ^p satisfies the following four basic properties:

(a) $\|x_1, \dots, x_n\|_p = 0$ if and only if $x_1, \dots, x_n$ are linearly dependent;

(b) $\|x_1, \dots, x_n\|_p$ is invariant under permutation;

(c) $\|\alpha x_1, x_2, \dots, x_n\|_p = |\alpha| \, \|x_1, x_2, \dots, x_n\|_p$ for any $\alpha \in \mathbb{R}$;

(d) $\|x_1 + x_1', x_2, \dots, x_n\|_p \le \|x_1, x_2, \dots, x_n\|_p + \|x_1', x_2, \dots, x_n\|_p$.

Further properties of this functional on ℓ^p can be found in [**1**]. See also [**2**, **5**], and the references therein, for related works.

Our theorem below relates the "volume" $V(x_1, \dots, x_n)$ defined by (3) and the n-norm $\|x_1, \dots, x_n\|_p$, which also represents a volume of the n-dimensional parallelepiped spanned by $x_1, \dots, x_n$ in ℓ^p.

We assume hereafter that $n \ge 2$.

Theorem 1. *Let $\{x_1, \dots, x_n\}$ be a linearly independent set of vectors in ℓ^p. For any permutation $(i_1, \dots, i_n)$ of $(1, \dots, n)$, define $V(x_{i_1}, \dots, x_{i_n})$ as in (3) by using the semi-inner product g in (1), with $x_1^* = x_{i_1}$ and so forth as in (2). Then we have*

$$V(x_{i_1}, \dots, x_{i_n}) \le (n!)^{1/p} \|x_1, \dots, x_n\|_p.$$

The following example illustrates the situation in ℓ^1. Let $x_1 = (1, 0, 0, \dots)$ and $x_2 = (1, 1, 0, \dots)$. Put $x_1^* = x_1$ and $x_2^* = x_2 - (x_2)_{x_1} = (0, 1, 0, \dots)$. Then we have $V(x_1, x_2) = \|x_1^*\|_1 \|x_2^*\|_1 = 1 \cdot 1 = 1$. But if we put $x_1^* = x_2$ and $x_2^* = x_1 - (x_1)_{x_2} = (\frac{1}{2}, -\frac{1}{2}, 0, \dots)$, then we have $V(x_2, x_1) = \|x_1^*\|_1 \|x_2^*\|_1 = 2 \cdot 1 = 2$. Meanwhile,

$$\|x_1, x_2\|_1 = \frac{1}{2}\sum_j \sum_k \left\| \begin{matrix} x_{1j} & x_{1k} \\ x_{2j} & x_{2k} \end{matrix} \right\| = \frac{1}{2}\left(\left| \begin{matrix} 1 & 0 \\ 1 & 1 \end{matrix} \right| + \left| \begin{matrix} 0 & 1 \\ 1 & 1 \end{matrix} \right| \right) = \frac{1}{2}(1+1) = 1.$$

Hence we see that $V(x_{i_1}, x_{i_2}) \le 2\|x_1, x_2\|_1$ for each permutation (i_1, i_2) of $(1, 2)$.

Proof of Theorem 1. Since $\|x_1, \ldots, x_n\|_p$ is invariant under permutation, it suffices for us to show that

$$V(x_1, \ldots, x_n) \le (n!)^{1/p}\|x_1, \ldots, x_n\|_p.$$

Recall that $V(x_1, \ldots, x_n) = \prod_{i=1}^{n} \|x_i^*\|$, where $x_1^*, \ldots, x_n^*$ is the left g-orthogonal sequence constructed from $x_1, \ldots, x_n$ (with $x_1^* = x_1$ and so forth as in (2)). From the construction of $x_1^*, \ldots, x_n^*$, we have

$$x_n^* = \frac{1}{\Gamma(x_1^*, \ldots, x_{n-1}^*)} \left| \begin{matrix} x_n & x_1^* & \ldots & x_{n-1}^* \\ g(x_1^*, x_n) & g(x_1^*, x_1^*) & \ldots & g(x_1^*, x_{n-1}^*) \\ \vdots & \vdots & \ddots & \vdots \\ g(x_{n-1}^*, x_n) & g(x_{n-1}^*, x_1^*) & \ldots & g(x_{n-1}^*, x_{n-1}^*) \end{matrix} \right|.$$

But $\Gamma(x_1^*, \ldots, x_{n-1}^*) = \prod_{i=1}^{n-1} \|x_i^*\|_p^2$, and so

$$\|x_n^*\|_p = \prod_{i=1}^{n-1} \|x_i^*\|_p^{-2} \left(\sum_{j_n} \left\| \begin{matrix} x_{nj_n} & x_{1j_n}^* & \ldots & x_{n-1,j_n}^* \\ g(x_1^*, x_n) & g(x_1^*, x_1^*) & \ldots & g(x_1^*, x_{n-1}^*) \\ \vdots & \vdots & \ddots & \vdots \\ g(x_{n-1}^*, x_n) & g(x_{n-1}^*, x_1^*) & \ldots & g(x_{n-1}^*, x_{n-1}^*) \end{matrix} \right\|^p \right)^{1/p}.$$

Hence, the "volume" $V(x_1, \ldots, x_n)$ is equal to

$$\prod_{i=1}^{n-1} \|x_i^*\|_p^{-1} \left(\sum_{j_n} \left\| \begin{matrix} x_{nj_n} & x_{1j_n}^* & \ldots & x_{n-1,j_n}^* \\ g(x_1^*, x_n) & g(x_1^*, x_1^*) & \ldots & g(x_1^*, x_{n-1}^*) \\ \vdots & \vdots & \ddots & \vdots \\ g(x_{n-1}^*, x_n) & g(x_{n-1}^*, x_1^*) & \ldots & g(x_{n-1}^*, x_{n-1}^*) \end{matrix} \right\|^p \right)^{1/p}.$$

By using properties of determinants, we find that $V(x_1, \ldots, x_n)$ is equal to

$$\left[\sum_{j_n} \left| \prod_{i=1}^{n-1} \|x_i^*\|_p^{-1} \right| \begin{matrix} g(x_1^*, x_1^*) & \ldots & g(x_{n-1}^*, x_1^*) & x_{1jn}^* \\ \vdots & \vdots & \ddots & \vdots \\ g(x_1^*, x_{n-1}^*) & \ldots & g(x_{n-1}^*, x_{n-1}^*) & x_{n-1,j_n}^* \\ g(x_1^*, x_n) & \ldots & g(x_{n-1}^*, x_n) & x_{nj_n} \end{matrix} \right\|^p \Bigg)^{1/p}.$$

Since $x_1^* = x_1$ and $g(x, y)$ is linear in y, it follows that $V(x_1, \ldots, x_n)$ is equal to

$$\left(\sum_{j_n} \left| \prod_{i=1}^{n-1} \|x_i^*\|_p^{-1} \begin{vmatrix} g(x_1^*, x_1) & \ldots & g(x_{n-1}^*, x_1) & x_{1jn} \\ \vdots & \vdots & \ddots & \vdots \\ g(x_1^*, x_{n-1}) & \ldots & g(x_{n-1}^*, x_{n-1}) & x_{n-1,j_n} \\ g(x_1^*, x_n) & \ldots & g(x_{n-1}^*, x_n) & x_{nj_n} \end{vmatrix} \right|^p \right)^{1/p}.$$

Now $g(x_i^*, x_k) = \|x_i^*\|_p^{2-p} \sum_{j_i} |x_{ij_i}|^{p-1} \mathrm{sgn}(x_{ij_i}) x_{kj_i}$, and we can take the sums out of the determinant, so that the above expression is dominated by

$$\left(\sum_{j_n} \left(\sum_{j_{n-1}} \cdots \sum_{j_1} \frac{|x_{n-1,j_{n-1}}|^{p-1}}{\|x_{n-1}^*\|_p^{p-1}} \cdots \frac{|x_{1j_1}|^{p-1}}{\|x_1^*\|_p^{p-1}} \left| \begin{vmatrix} x_{1j_1} & \ldots & x_{1j_n} \\ \vdots & \ddots & \vdots \\ x_{nj_1} & \ldots & x_{nj_n} \end{vmatrix} \right| \right)^p \right)^{1/p}.$$

By HÖLDER's inequality (applied to the multiple series inside the inner square brackets), the last expression is dominated by

$$\left(\sum_{j_n} \sum_{j_{n-1}} \cdots \sum_{j_1} \left| \begin{vmatrix} x_{1j_1} & \ldots & x_{1j_n} \\ \vdots & \ddots & \vdots \\ x_{nj_1} & \ldots & x_{nj_n} \end{vmatrix} \right|^p \right)^{1/p},$$

which is equal to $(n!)^{1/p} \|x_1, \ldots, x_n\|_p$. This proves our theorem.

3. CONCLUDING REMARKS

Unlike in inner product spaces, we generally do not have an analogue of HADAMARD's inequality (see, e.g., [**6**, p. 597])

$$V(x_1, \ldots, x_n) \le \prod_{i=1}^n \|x_i\|.$$

For a counterexample, take $x_1 = (1, 2, 0, \ldots)$ and $x_2 = (2, -1, 0, \ldots)$ in ℓ^1. Then one may check that $V(x_1, x_2) = V(x_2, x_1) = 3 \cdot \frac{10}{3} > \|x_1\|_1 \|x_2\|_1$. (This adds a reason why we write the word "volume" between quotation marks for $V(x_1, \ldots, x_n)$.)

It is worth noting, however, that the analogue of Hadamard's inequality is satisfied particularly by the n-norm $\|\cdot, \ldots, \cdot\|_1$ on ℓ^1. Indeed, the inequality

$$\|x_1, \ldots, x_n\|_1 \le \prod_{i=1}^n \|x_i\|_1$$

holds for every $x_1, \ldots, x_n$ in ℓ^1 (see [**1**]). Hence the n-norm $\|\cdot, \ldots, \cdot\|_1$ has the desirable properties for volumes of n-dimensional parallelepipeds in ℓ^1.

The reader might also wonder why we do not define the volume of the n-dimensional parallelepiped spanned by $x_1, \dots, x_n$ in X to be

$$V(x_1, \dots, x_n) := \sqrt{\Gamma(x_1, \dots, x_n)},$$

instead of (3). Although $\Gamma(x_1, \dots, x_n) = \det\big(g(x_i, x_j)\big)$ is invariant under permutation, there are a few problems with this formula. First, $\Gamma(x_1, \dots, x_n)$ may be negative when $n \geq 3$. For example, take $x_1 = (1, 2, -1/10, 0, \dots)$, $x_2 = (2, 1, 1/10, 0, \dots)$, and $x_3 = (1, -1, 1, 0, \dots)$ in ℓ^1. Then one may check that $\Gamma(x_1, x_2, x_3) < 0$. Next, for $n = 2$, we can have $\Gamma(x_1, x_2) = 0$ even though x_1 and x_2 are linearly dependent. For example, take $x_1 = (1, 2, 0, \dots)$ and $x_2 = (2, 1, 0, \dots)$ in ℓ^1. Clearly x_1 and x_2 are linearly independent. But one may check that $g(x_i, x_j) = 9$ for $i, j = 1, 2$, and so $\Gamma(x_1, x_2) = \begin{vmatrix} 9 & 9 \\ 9 & 9 \end{vmatrix} = 0$. (This explains why we require $\Gamma(x_1, \dots, x_n) \neq 0$ when we define the g-orthogonal projection on the subspace $S = \mathrm{span}\{x_1, \dots, x_n\}$.)

One should also note that the analogue of Hadamard's inequality is not satisfied by $|\Gamma|$, that is, the inequality

$$|\Gamma(x_1, \dots, x_n)| \leq \prod_{i=1}^{n} \|x_i\|^2$$

does not hold. For a counterexample, take $x_1 = (1, 2, 0, \dots)$ and $x_2 = (2, -1, 0, \dots)$ in ℓ^1. Then we have $|\Gamma(x_1, x_2)| = 90 > \|x_1\|_1^2 \|x_2\|_1^2$. Nevertheless, we have the following result for Γ. (We leave its proof to the reader.)

Theorem 2. *The inequality*

$$|\Gamma(x_1, \dots, x_n)| \leq (n!)^{1/p} \|x_1, \dots, x_n\|_p \prod_{i=1}^{n} \|x_i\|_p$$

holds for every $x_1, \dots, x_n$ in ℓ^p.

Acknowledgement. The research is supported by the Directorate General of Higher Education of Republic of Indonesia through Hibah Pekerti I Program, 2003/2004.

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(Received July 15, 2004)

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